DEPARTMENT OF MATHEMATICS

Mathematics for Computer Science

(COURSE CODE 23MATS31)

Module-2

Markov's Chain and Joint Probability Distribution

Probability vectors, stochastic matrices, fixed point matrices, regular stochastic matrices, Markov's Chains, higher transition probabilities, stationary distribution of regular Markov's Chains. Joint Probability distribution of two discrete random variables. Expectations, covariance and correlation.

MARKOV CHAIN:

Introduction:

Vector:

A vector is an n-tuple of numbers $u = (u_1,\ u_2,\ u_3, \dots \dots, u_n)$, where u_i are called the components of u.

If all the $u_i = 0$, then u is called the zero vector.

If $u = (u_1,\ u_2,\ u_3, \dots \dots, u_n)$ is a vector and k is a real number, then the product

$ku = (ku_1, ku_2, ku_3, \dots \dots, ku_n)$ is called scalar multiple of u by a scalar k.

Fixed vector or Fixed point:

If A is an n-square matrix (matrix of order $n \times n$) and u is a vector with n components $(u \neq 0)$ such that uA = u, then u is called a fixed vector or fixed point of A.

In this case for any scalar $k \neq 0$, we have (ku)A = k(uA)=ku

Example:

Let $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ be a matrix and u = (2, -1) be a vector, then $uA = (2, -1)\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = (2, -1) = u$

$\therefore\ u = (2, -1)$ is a fixed point of A.

Again, 2u = 2(2, -1) = (4, -2) and (2u)A = (4, -2) $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ = (4, -2) = 2u.

Probability Vector:

A vector $u = (u_1, u_2, u_3, \dots\dots, u_n)$ is called a probability vector if the components are non-negative and their sum is 1. i.e. $u_i \geq 0$ and $\sum_{i=1}^{n} u_i = 1$.

Example:

1. $u = \left(\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}\right)$ is a probability vector

2. $v = \left(\frac{3}{4}, 0, -\frac{1}{4}, \frac{1}{2}\right)$ is not a probability vector. $\because -\frac{1}{4}$ is negative.

3. $w = \left(\frac{3}{4}, \frac{1}{2}, 0, \frac{1}{4}\right)$ is not a probability vector. $\because$ sum is >1.

Stochastic matrix:

A square matrix $P = (p_{ij})$ is called a stochastic matrix if each of its row is a probability vector.

Example:

1. $A = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$ is a stochastic matrix.

2. $B = \begin{pmatrix} \frac{1}{2} & 0 & \frac{2}{3} \\ \frac{3}{4} & \frac{1}{2} & \frac{-1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ and $C = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ are not stochastic matrices.

Theorems:

1. Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ be a stochastic matrix and $u = (u_1, u_2, u_3)$ be a probability vector, then uA is also a probability vector.

2. If $A = (a_{ij})$ is a stochastic matrix of order n and $u = (u_1, u_2, \dots\dots, u_n)$ is a probability vector, then uA is also a probability vector.

3. If A and B are stochastic matrices, then the product AB is also a stochastic matrix. Therefore, in particular, all powers A^n are stochastic matrices.

Regular Stochastic Matrix:

A stochastic matrix P is said to be regular if all the entries of some power P^m are positive.

Example:

1. The stochastic matrix $A = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is regular, since

$A^2 = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$. We observe that all entries are positive.

2. The stochastic matrix $A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is not regular, since $A^2 = \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$,

$A^3 = \begin{pmatrix} 1 & 0 \\ \frac{7}{8} & \frac{1}{8} \end{pmatrix}$, $A^4 = \begin{pmatrix} 1 & 0 \\ \frac{15}{16} & \frac{1}{16} \end{pmatrix}$. we observe that every power A^m will have 1.

and 0 in the first row.

Theorem:

Let P be a regular stochastic matrix, then

(i) P has a unique fixed probability vector t and the components of t are all positive.

(ii) The sequence $P, P^2, P^3, \ldots\ldots$, of powers of P approaches the matrix T whose rows are each the fixed point t.

(iii) If p is any probability vector, then the sequence of vectors $pP, pP^2, pP^3, \ldots\ldots$ approaches to the fixed point t.

Note:

P^n approaches T means that each entry of P^n approaches the corresponding entry of T, and pP^n approaches t means that each component of pP^n approaches the corresponding components of t.

Problems:

1. Find the unique fixed probability vector of the regular stochastic matrix $A = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

What matrix does A^n approach?

Solution:

Given $A = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Let $t = (x, 1-x)$ be the probability vector such that $tA = t$

$$\therefore (x, 1-x)\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (x, 1-x).$$

$\therefore\ \frac{3}{4}x + (1-x)\left(\frac{1}{2}\right) = x$ and $\frac{1}{4}x + \left(\frac{1}{2}\right)(1-x) = 1-x.$

$\therefore\ 3x + 2 - 2x = 4x$ and $x + 2 - 2x = 4 - 4x.$

$\therefore\ x = \frac{2}{3}$ and $x = \frac{2}{3}.$

$\therefore\ t = \left(\frac{2}{3}, 1 - \frac{2}{3}\right) = \left(\frac{2}{3}, \frac{1}{3}\right)$ is the required probability vector.

The matrix A^n approaches the matrix T whose rows are each the fixed point t. i.e., $T = \left(\frac{2}{3}, \frac{1}{3}\right)$.

2. Show that $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is a regular stochastic matrix. Also find the associated unique fixed probability vector.

Solution: Given $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P^3 = P^2.P = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$P^4 = P^3.P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$P^5 = P^4.P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

We observe that in P^5 all the entries are positive.

Thus P is a regular stochastic matrix.

Let t = (x, y, 1-x-y) be the unique fixed probability vector then tP = t

$$\therefore (x, y, 1-x-y)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} = (x, y, 1-x-y)$$

$\frac{1}{2} - \frac{x}{2} - \frac{y}{2} = x$, $x + \frac{1}{2} - \frac{x}{2} - \frac{y}{2} = y$, $y = 1 - x - y$

$1 - x - y = 2x$, $2x + 1 - x - y = 2y$, $2y = 1 - x$

$y = 1 - 3x$ $\qquad$ $x = 3y - 1$

$y = 1 - 3(3y - 1)$ $\qquad$ $x = 3\left(\frac{2}{5}\right) - 1$

$y = 1 - 9y + 3$ $\qquad$ $\therefore x = \frac{1}{5}$

$10y = 4$

$\therefore y = \frac{4}{10} = \frac{2}{5}$

$\therefore t = \left(\frac{1}{5}, \frac{2}{5}, 1 - \frac{1}{5} - \frac{2}{5}\right) = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$ is the unique fixed probability vector.

3. (i) Show that the vector u = (b, a) is a fixed point of the general 2×2 stochastic matrix $P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$.

(ii) Use the result of (i) to find the unique fixed probability vector of each of the following matrices: $A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ and $C = \begin{pmatrix} 0.7 & 0.3 \\ 0.8 & 0.2 \end{pmatrix}$.

Solution:

Given $P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$ and u = (b, a).

(i) $uP = (b, a)\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} = (b - ab + ab, ab + a - ab) = (b, a) = u.$ $\therefore uP = u.$

$\therefore$ u is a fixed point of P.

(ii) Given $A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$. $\qquad \therefore a = \frac{2}{3}$ and $b = 1$.

$\therefore u = (b, a) = \left(1, \frac{2}{3}\right)$ is a fixed point of A.

Multiply u by 3. Then (3, 2) is the fixed point of A. Now 3 + 2 = 5.

Divide this vector by 5. $\therefore$ $\left(\frac{3}{5}, \frac{2}{5}\right)$ is the required unique fixed probability vector by A.

Given $B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$. $\therefore$ $a = \frac{1}{2}$ and $b = \frac{2}{3}$.

$\therefore u = (b,\ a) = \left(\frac{2}{3}, \frac{1}{2}\right)$ is a fixed point of B.

Multiply u by 6. Then (4, 3) is the fixed point of B. Now 4 + 3 = 7.

Divide this vector by 7 $\therefore$ $\left(\frac{4}{7}, \frac{3}{7}\right)$ is the required unique fixed probability vector by B.

Given $C = \begin{pmatrix} 0.7 & 0.3 \\ 0.8 & 0.2 \end{pmatrix}$. $\therefore$ $a = 0.3$ and $b = 0.8$.

$\therefore$ $u = (b,\ a) = (0.8,\ 0.3) = \left(\frac{8}{10}, \frac{3}{10}\right)$ is a fixed point of C.

Multiply u by 10. Then (8, 3) is the fixed point of C. Now 8 + 3 = 11.

Divide this vector by 11. $\therefore$ $\left(\frac{8}{11}, \frac{3}{11}\right)$ is the required unique fixed probability vector by c

4. Find the unique fixed probability vector of the regular stochastic matrix

$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$. What matrix does P^n approach?

Solution:

Let $t = (x, y, 1 - x - y)$ be the unique fixed probability vector then $t P = t$.

$$\therefore \quad (x, y, 1-x-y)\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} = (x, y, 1-x-y).$$

$\therefore \quad \frac{1}{2}x + \frac{1}{2}y = x, \qquad \frac{1}{4}x + 1 - x - y = y, \qquad \frac{1}{4}x + \frac{1}{2}y = 1 - x - y.$

$\therefore \quad x + y = 2x, \qquad x + 4 - 4x - 4y = 4y, \qquad x + 2y = 4 - 4x - 4y.$

$\therefore \quad y = x, \qquad 3x + 8y = 4, \qquad 5x + 6y = 4.$

$\therefore \quad 3x + 8x = 4. \quad \therefore \quad 11x = 4. \quad \therefore \; x = \frac{4}{11}. \quad \therefore \quad y = \frac{4}{11}.$

$\therefore \quad t = \left(\frac{4}{11}, \frac{4}{11}, 1 - \frac{4}{11} - \frac{4}{11}\right) = \left(\frac{4}{11}, \frac{4}{11}, \frac{3}{11}\right)$ is the unique fixed probability vector.

The matrix P^n approaches the matrix T whose each row is the fixed point t where

$$T = \begin{pmatrix} \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \end{pmatrix}.$$

5. Show that $P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ is a regular stochastic matrix. Also find the associated unique fixed probability vector. What matrix does P^n approach?

Solution:

Given $P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$

$$\therefore \quad P^2 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{12} & \frac{23}{36} & \frac{7}{6} \\ \frac{1}{9} & \frac{5}{9} & \frac{1}{3} \end{pmatrix}.$$

We observe that in P^2 all the entries are positive.

Thus P is a regular stochastic matrix.

Let $t = (x, y, 1-x-y)$ be the unique fixed probability vector then $tP = t$.

$$\therefore \quad (x, y, 1-x-y)\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = (x, y, 1-x-y).$$

$\therefore \quad \frac{y}{6} = x, \qquad x + \frac{y}{2} + \frac{2}{3} - \frac{2x}{3} - \frac{2y}{3} = y, \qquad \frac{y}{3} + \frac{1}{3} - \frac{x}{3} - \frac{y}{3} = 1 - x - y.$

$\therefore \quad y = 6x, \qquad \frac{x}{3} + \frac{2}{3} - \frac{y}{6} = y. \quad \therefore \quad \frac{x}{3} - \frac{7y}{6} = -\left(\frac{2}{3}\right). \quad \therefore \quad \frac{x}{3} - \frac{7(6x)}{6} = -\left(\frac{2}{3}\right).$

$\therefore \quad -\left(\frac{20x}{3}\right) = -\left(\frac{2}{3}\right). \quad \therefore \quad x = \frac{1}{10}. \quad \therefore \quad y = 6\left(\frac{1}{10}\right). \quad \therefore \quad y = \frac{3}{5}.$

$\therefore \quad t = \left(\frac{1}{10}, \frac{3}{5}, 1 - \frac{1}{10} - \frac{3}{5}\right) = \left(\frac{1}{10}, \frac{3}{5}, \frac{3}{10}\right)$ is the unique fixed probability vector.

The matrix P^n approaches the matrix T whose each row is the fixed point t where

$$T = \begin{pmatrix} \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \end{pmatrix}.$$

6. Find the unique fixed probability vector of the regular stochastic matrix $P = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$.

What matrix does P^n approach?

Solution: Let t = (x, 1-x) be the probability vector such that tP = t

$$\therefore (x, 1-x)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (x, 1-x)$$

$\frac{1}{2} - \frac{x}{2} = x$ and $x + \frac{1}{2} - \frac{x}{2} = 1 - x$

$1 - x = 2x$ $\quad$ $2x + 1 - x = 2 - 2x$

$3x = 1$ $\quad$ $3x = 1$

$x = \frac{1}{3}$ $\quad$ $x = \frac{1}{3}$

$\therefore t = \left(\frac{1}{3}, 1 - \frac{1}{3}\right) = \left(\frac{1}{3}, \frac{2}{3}\right)$ is the unique fixed probability vector.

The matrix P^n approaches the matrix T whose each row is the fixed point t where $T = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix}$.

HOME WORK

1. Find the associated unique fixed probability vector of the regular stochastic matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$. What matrix does P^n approach?

2. Find the unique fixed probability vector for the regular stochastic matrix $\begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

3. Show that $P = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$ is a regular stochastic matrix. Also find the associated unique fixed probability vector.

Markov Chains:

Consider a sequence of trials (experiments) whose outcomes, say $X_1,\ X_2,\ X_{3,}\ \dots \dots ..$ satisfy the following two properties:

(i) Each outcomes belongs to a finite set of outcomes $\{a_1, a_2, \dots \dots a_m\}$ called the state space of the system. a_i is called a state.

(ii) The outcome of any trial depends at most upon the outcome of the immediately preceding trial and not upon any other previous outcome; with each pair of states (a_i, a_j), there is given the probability p_{ij} that a_j occurs immediately after a_i occurs.

Such a stochastic process is called a (finite) Markov chain.

The number p_{ij} are called the transition probabilities. p_{ij}'s can be arranged in a matrix as

$$P = \begin{pmatrix} p_{11} & p_{12} \dots . \dots \dots \dots . p_{1m} \\ p_{21} & p_{22} \dots . \dots \dots \dots . p_{2m} \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ p_{m1} & p_{m2} \dots \dots \dots \dots p_{mm} \end{pmatrix}$$ is called the transition matrix.

Thus with each state a_i there corresponds the i^{th} row $(p_{i1},\ p_{i2}, \dots \dots, p_{im})$ of the transition matrix P.

If the system is in state a_i, then the i^{th} row $(p_{i1}, p_{i2}, \dots \dots, p_{im})$ represents the probabilities of all possible outcomes of the next trial and so it is a probability vector. Thus the transition matrix P of a Markov chain is a stochastic matrix.

Higher transition probabilities:

The entry p_{ij} in the transition matrix P of a Markov chain is the probability that the system changes from the state a_i to the state a_j in one step $(a_i \to a_j)$. Similarly, the probability denoted by ${p_{ij}}^{(n)}$ that the system changes from the state a_i to the state a_j in exactly n steps. i.e., $(a_i \to a_{k_1} \to a_{k_2} \to \cdots \to a_{k_{n-1}} \to a_j)$.

Theorem:

Let P be the transition matrix of a Markov chain process, then the n-step transition matrix is equal to the n^{th} power of P. i.e., $P^{(n)} = P^n$.

NOTE:

1. At some arbitrary time, the probability that the system is in state a_i is p_i. We denote

these probabilities by the probability vector $\mathbf{P} = (\mathbf{P_1}, \mathbf{P_2}, \dots \dots, \mathbf{P_m})$ which is called the probability distribution of the system at that time.

In particular, let $\mathbf{P}^{(0)} = (\mathbf{P_1}^{(0)}, \ \mathbf{P_2}^{(0)}, \dots \dots, \mathbf{P_m}^{(0)})$ denote the initial probability distribution, then $\mathbf{P}^{(n)} = (\mathbf{P_1}^{(n)}, \mathbf{P_2}^{(n)}, \dots \dots, \mathbf{P_m}^{(n)})$ denote the $\mathbf{n^{th}}$ step probability distribution. i.e., the probability after the first n steps.

2. $\mathbf{p^{(1)} = p^{(0)}P, \ \ p^{(2)} = p^{(1)}P = p^{(0)}P^2, \ \ p^{(3)} = p^{(2)}P = p^{(0)}P^3, \dots \dots \dots, p^{(n)} = p^{(0)}P^n.}$

Stationary Distribution of Regular Markov chains:

Theorem:

If the transition matrix P of a Markov chain is regular, then in the long run, the probability that the state a_j occurs in approximately equal to the component t_j of the unique fixed probability vector t of P.

Stationary distribution or steady state:

Every sequence of probability distribution approaches the fixed probability vector t of P called the stationary distribution or steady state of the Markov chain where P is the transition matrix of the Markov chain.

Problems:

1. Given the transition matrix $P = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ with initial probability distribution $p^{(0)} = \left(\frac{1}{3}, \frac{2}{3}\right)$. Define and find (i) $p_{21}^{(3)}$, (ii) $p^{(3)}$, (iii) $p_2^{(3)}$.

Solution:

(i) $p_{21}^{(3)}$ is the probability of moving from state a_2 to state a_1 in 3 steps and is obtained from the 3-step transition matrix P^3

Now, $P^2 = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$, $P^3 = P^2P = \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{7}{8} & \frac{1}{8} \end{pmatrix}$

$\therefore$ $p_{21}^{(3)}$ = entry of the second row first column of $P^3 = \frac{7}{8}$

(ii) $p^{(3)}$ is the probability distribution of the system after 3 steps.

$$p^{(1)} = p^{(0)}P = \left(\frac{1}{3}, \frac{2}{3}\right)\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{2}{3}, \frac{1}{3}\right).$$

$$p^{(2)} = p^{(1)}P = \left(\frac{2}{3}, \frac{1}{3}\right)\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{5}{6}, \frac{1}{6}\right).$$

$p^{(3)} = p^{(2)}P = \left(\frac{5}{6}, \ \frac{1}{6}\right)\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{11}{12}, \ \frac{1}{12}\right).$

Otherwise, $p^{(3)} = p^{(0)}P^3 = \left(\frac{1}{3}, \ \frac{2}{3}\right)\begin{pmatrix} 1 & 0 \\ \frac{7}{8} & \frac{1}{8} \end{pmatrix} = \left(\frac{11}{12}, \ \frac{1}{12}\right)$

(iii) $p_2^{(3)}$ is the probability that the process is in the start a_2 after 3 steps. i.e. the second component of $p^{(3)}$. $\therefore \ p_2{}^{(3)} = \frac{1}{12}$.

2. Given the transition matrix $P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and the initial probability distribution $p^{(0)} = \left(\frac{2}{3}, \ 0, \ \frac{1}{3}\right)$. Find (i) $p_{32}^{(2)}$ *and* $p_{13}^{(2)}$, (ii) $p^{(4)}$ *and* $p_3^{(4)}$, (iii) the vector that $p^{(0)}P^n$ approaches, (iv) the matrix that P^n approaches.

Solution:

(i) $P^2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$. $\therefore \ p_{32}^{(2)} = \frac{1}{2}$ *and* $p_{13}^{(2)} = 0$.

(ii) $P^4 = P^2.P^2 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{9}{16} & \frac{3}{16} \\ \frac{5}{16} & \frac{9}{16} & \frac{1}{8} \\ \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \end{pmatrix}$.

$$p^{(4)} = p^{(0)}P^4 = \left(\frac{2}{3}, \ 0, \ \frac{1}{3}\right)\begin{pmatrix} \frac{1}{4} & \frac{9}{16} & \frac{3}{16} \\ \frac{5}{16} & \frac{9}{16} & \frac{1}{8} \\ \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \end{pmatrix} = \left(\frac{1}{4}, \ \frac{7}{12}, \ \frac{1}{6}\right).$$

$\therefore \ p_3^{(4)} = \frac{1}{6}$

(iii) $p^{(0)}P^n$ approaches the unique fixed probability vector t of P.

Let t = (x, y, $1 - x - y$) be the fixed point of P then t P = t.

$\therefore (x, \ y, \ 1 - x - y)\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix} = (x, \ y, \ 1 - x - y).$

$\frac{y}{2} = x \qquad , \qquad \frac{x}{2} + \frac{y}{2} + 1 - x - y = y \qquad , \qquad \frac{x}{2} = 1 - x - y$

$y = 2x \qquad , \qquad x + y + 2 - 2x - 2y = 2y \qquad , \qquad x = 2 - 2x - 2y$

$y = 2\left(\frac{2}{7}\right) \qquad\qquad 2 - x = 3y \qquad\qquad 3x = 2 - 2y$

$\therefore y = \frac{4}{7} \qquad\qquad 2 - x = 3(2x)$

$$2 = 7x$$

$$\therefore x = \frac{2}{7}$$

$\therefore t = \left(\frac{2}{7}, \frac{4}{7}, 1 - \frac{2}{7} - \frac{4}{7}\right) = \left(\frac{2}{7}, \frac{4}{7}, \frac{1}{7}\right)$

$\therefore p^{(0)}P^n$ approaches the unique fixed probability vector $t = \left(\frac{2}{7}, \frac{4}{7}, \frac{1}{7}\right)$

(iv) P^n approaches the matrix T whose rows are each the fixed probability vector of P.

$\therefore P^n$ approaches $\begin{pmatrix} \frac{2}{7} & \frac{4}{7} & \frac{1}{7} \\ \frac{2}{7} & \frac{4}{7} & \frac{1}{7} \\ \frac{2}{7} & \frac{4}{7} & \frac{1}{7} \end{pmatrix}$

2. A man either drives his car or catches a train to work each day. Suppose he never goes by train two days in a row, but if he drives to work, then the next day he is just as likely to drive again as he is to travel by train. Find the transition matrix for the chain of the mode of transport he uses. If he drives on the first day of work, find the probability that (i) car is used (ii) train is used, on the fifth day (iii) In long run, how often does he take train or drive to work.

Solution:

The state space of the system {t(train), d(drive)}. This stochastic process is a Markov chain since the outcome on any day depends only on what happened the preceding day. The transition matrix of the Markov chain is

$$\mathrm{P} = \begin{matrix} & \mathrm{t} & \mathrm{d} \\ \mathrm{t} & 0 & 1 \\ \mathrm{d} & \frac{1}{2} & \frac{1}{2} \end{matrix}.$$

$\therefore \mathrm{P}^2 = \mathrm{PP} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$

$\therefore P^4 = P^2.P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{pmatrix}$

Suppose the man drives his car on the first day of work, the initial probability distribution is $p^{(0)} = (0,1)$. To find the probability on the fifth day, i.e., after 4 days, we take $p^{(4)} = p^{(0)}P^4$. $\therefore p^{(4)} = p^{(0)}P^4 = (0,1)\begin{pmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{pmatrix} = \left(\frac{5}{16}, \frac{11}{16}\right)$ is the probability distribution of the mode of the transport on fifth day. $\therefore$ On the fifth day,

(i) The probability that he used the car = $\frac{11}{16}$.

(ii) The probability that he used the train = $\frac{5}{16}$.

(iii) Let t = (x, 1− x) be the probability vector such that t P = t.

$\therefore \ (x, 1-x)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (x, 1-x). \quad \therefore \ \frac{1}{2} - \frac{x}{2} = x$ and $x + \frac{1}{2} - \frac{x}{2} = 1 - x.$

$\therefore \ 1 - x = 2x$ and $2x + 1 - x = 2 - 2x.$

$\therefore \ 3x = 1$ and $3x = 1. \quad \therefore \ x = \frac{1}{3}.$

$\therefore \ t = \left(\frac{1}{3}, \ 1 - \frac{1}{3}\right) = \left(\frac{1}{3}, \ \frac{2}{3}\right).$

In the long run, P approaches the matrix T, each of its row is a unique fixed point $t = \left(\frac{1}{3}, \frac{2}{3}\right)$.

$\therefore$ In long run, he use the train $\frac{1}{3}$ of the time and use the car $\frac{2}{3}$ of the time.

Three boys A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C just as likely to throw the ball to B as to A. If C was the first person to throw the ball, find the probabilities that for the fourth throw (after three throws). (i) A has the ball, (ii) B has the ball, (iii) C has the ball, and (iv) In long run how often does each throw the ball.

Solution:

We take the state space of the system as {A, B, C}.

$$\therefore \text{ The transition matrix of the Markov chain is } P = \begin{array}{c} \\ A \\ B \\ C \end{array}\!\!\begin{array}{c} \begin{matrix} \text{A} & \text{B} & \text{C} \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

Suppose C was the first person with the ball, the initial probability distribution is $p^{(0)} = (0, 0, 1)$. Then

$$p^{(1)} = p^{(0)}P = (0,\ 0,\ 1)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \left(\frac{1}{2},\ \frac{1}{2},\ 0\right).$$

$$p^{(2)} = p^{(1)}P = \left(\frac{1}{2},\ \frac{1}{2},\ 0\right)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \left(0,\ \frac{1}{2},\ \frac{1}{2}\right).$$

$$p^{(3)} = p^{(2)}P = \left(0,\ \frac{1}{2},\ \frac{1}{2}\right)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \left(\frac{1}{4},\ \frac{1}{4},\ \frac{1}{2}\right). \quad \text{OR}$$

$$P^2 = P.P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

$$P^3 = P^2P = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

$$p^{(3)} = p^{(0)}P^3 = (0, 0, 1)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right).$$

Thus after three throws, the probability that A has the ball is $\frac{1}{4}$ that B has the ball is $\frac{1}{4}$ and C has the ball is $\frac{1}{2}$.

i.e., ${P_A}^{(3)} = \frac{1}{4}$, ${P_B}^{(3)} = \frac{1}{4}$ *and* ${P_C}^{(3)} = \frac{1}{2}$.

Let t = (x, y, $1 - x - y$) be the unique fixed probability vector then t P = t.

$\therefore\ (x, y, 1-x-y)\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix} = (x, y, 1-x-y).$

$\therefore\ \frac{1}{2}-\frac{x}{2}-\frac{y}{2}=x, \qquad x+\frac{1}{2}-\frac{x}{2}-\frac{y}{2}=y, \qquad y=1-x-y.$

$\therefore\ 1-x-y=2x, \qquad 2x+1-x-y=2y, \qquad 2y=1-x.$

$\therefore\ y=1-3x, \qquad x=3y-1.$

$\therefore\ y=1-3(3y-1). \quad \therefore\ y=1-9y+3. \quad \therefore\ 10y=4. \quad \therefore y=\frac{4}{10}=\frac{2}{5}.$

$\therefore\ x=3\left(\frac{2}{5}\right)-1. \quad \therefore\ x=\frac{1}{5}.$

$\therefore\ t=\left(\frac{1}{5}, \frac{2}{5}, 1-\frac{1}{5}-\frac{2}{5}\right)=\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right).$

$\therefore$ In the long run, A will be thrown the ball $\frac{1}{5}$ times. i.e., 20% of the time, B will be thrown the ball $\frac{2}{5}$ times. i.e., 40% of the time and C will be thrown the ball $\frac{2}{5}$ times. i.e., 40% of the time.

5. A student's study habits are as follows. If he studies one night, he is 70% sure not to study the next night. On the other hand, if he does not study one night, he is 60% sure not to study the next night as well. In the long run, how often does he study?

Solution:

The state space is {S(studying), T(not studying)}. The transition matrix is $P = \begin{matrix} & S & T \\ S & 0.3 & 0.7 \\ T & 0.4 & 0.6 \end{matrix}$.

Let t be the unique fixed point given by t = (x, $1-x$), then t P = t.

$\therefore\ (x, 1-x)\begin{pmatrix}0.3 & 0.7\\ 0.4 & 0.6\end{pmatrix}=(x, 1-x).$

$\therefore\ 0.3x+0.4-0.4x=x \quad$ and $\quad 0.7x+0.6-0.6x=1-x.$

$\therefore\ 0.4=1.1x. \quad \therefore\ x=\frac{0.4}{1.1}=\frac{4}{11}. \quad \therefore\ t=\left(\frac{4}{11}, 1-\frac{4}{11}\right)=\left(\frac{4}{11}, \frac{7}{11}\right).$

$\therefore$ In the long run the student studies $\frac{4}{11}$ of the time.

6. A psychologist makes the following assumptions concerning the behavior of mice subjected to a particular feeding schedule. For any particular trial 80% of the mice that went right in the previous experiment will go right in this trial, and 60% of those mice that went left in the previous experiment will go right in this trial. If 50% went right in the first trial, what would he predict for (i) the second trial, (ii) the third trial,

(iii) the thousandth trial?

Solution:

The state space is {R(right), L(left)}. The transition matrix is $P = \begin{matrix} & R & L \\ R & 0.8 & 0.2 \\ L & 0.6 & 0.4 \end{matrix}$.

Since 50% went right in the first trial, the initial probability vector is $p^{(0)} = (0.5,\ 0.5)$.

(i) $p^{(1)} = p^{(0)}P = (0.5, 0.5)\begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix} = (0.7,\ 0.3)$

∴ In the second trial that 70% of the mice will go right and 30% will go left.

(ii) $p^{(2)} = p^{(1)}P = (0.7,\ 0.3)\begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix} = (0.74,\ 0.26)$

∴ In the third trial he predict that 74% of the mice will go right and 26% will go left.

We assume that the probability distribution for the thousandth trial is essentially the stationary distribution of the Markov chain. i.e., the unique fixed probability vector t of the transition matrix P.

Let t = (x, $1 - x$) be the fixed point of P then t P = t.

∴ $(x, 1 - x)\begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix} = (x, 1 - x)$.

∴ $0.8x + 0.6 - 0.6x = x$ and $0.2x + 0.4 - 0.4x = 1 - x$.

∴ $0.6 = 0.8x$

∴ $x = \frac{0.6}{0.8} = \frac{3}{4}$. ∴ $t = \left(\frac{3}{4}, 1 - \frac{3}{4}\right) = \left(\frac{3}{4}, \frac{1}{4}\right) = (0.75,\ 0.25)$.

∴ He predicts that in the thousandth trial, 75% of the mice will go to the right and 25% will go to the left.

7. A salesman's territory consists of three cities A, B and C. He never sells in the same city on successive days. If he sells in city A, then the next day he sells in city B. However, if he sells in either B or C, then the next day he is twice as likely to sell in city A as in the other city. In the long run how often does he sell in each of the cities?

Solution:

The state space is {A, B, C}. The transition matrix of the Markov chain is

$$P = \begin{matrix} \\ A \\ B \\ C \end{matrix} \begin{matrix} \begin{matrix} A & B & C \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} \end{matrix}.$$

Let t = (x, y, $1 - x - y$) be the unique fixed probability vector then t P = t.

$$\therefore \ (x, y, 1 - x - y) \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} = (x, y, 1 - x - y).$$

$$\therefore \ \frac{2y}{3} + \frac{2}{3} - \frac{2x}{3} - \frac{2y}{3} = x, \quad x + \frac{1}{3} - \frac{x}{3} - \frac{y}{3} = y, \quad \frac{y}{3} = 1 - x - y.$$

$$\therefore \ 2y + 2 - 2x - 2y = 3x, \quad 3x + 1 - x - y = 3y, \quad y = 3 - 3x - 3y$$

$$\therefore \ 2 = 5x. \quad 2x + 1 = 4y.$$

$$\therefore \ x = \frac{2}{5}. \quad \therefore \ 2\left(\frac{2}{5}\right) + 1 = 4y. \quad \therefore \ 4y = \frac{9}{5}. \quad \therefore \ y = \frac{9}{20}.$$

$$\therefore \quad t = \left(\frac{2}{5}, \frac{9}{20}, 1 - \frac{2}{5} - \frac{9}{20}\right) = \left(\frac{2}{5}, \frac{9}{20}, \frac{3}{20}\right) = (0.40, \ 0.45, \ 0.15).$$

$\therefore$ In the long run he sells 40% of the time in city A, 45% of the time in city B and 15% of the time in city C.

8. Two boys B_1 and B_2 and two girls G_1 and G_2 are throwing ball from one to the other. Each boy throws the ball to the other boy with probability $\frac{1}{2}$ and to each girl with Probability $\frac{1}{4}$. On the other hand, each girl throws the ball to each boy with probability $\frac{1}{2}$ and never to the other girl. In the long run how often does each receive the ball?

Solution:

The state space is $\{B_1, B_2, G_1, G_2\}$. The transition matrix of the markov chain is

$$P = \begin{matrix} \\ B_1 \\ B_2 \\ G_1 \\ G_2 \end{matrix} \begin{matrix} \begin{matrix} B_1 & B_2 & G_1 & G_2 \end{matrix} \\ \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} \end{matrix}$$

Let $u = (x, y, z, w)$ be the fixed vector of P then u P = u.

$$\therefore \quad (x, y, z, w)\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} = (x, y, z, w).$$

$\therefore \ \frac{y}{2} + \frac{z}{2} + \frac{w}{2} = x, \quad \frac{x}{2} + \frac{z}{2} + \frac{w}{2} = y, \quad \frac{x}{4} + \frac{y}{4} = z, \quad \frac{x}{4} + \frac{y}{4} = w.$

$\therefore \ y + z + w = 2x \to (1), \ \ x + z + w = 2y \to (2), \ \ x + y = 4z \to (3), \ \ x + y = 4w \to (4).$

$(3) - (4) \Longrightarrow 4\text{z} - 4\text{w} = 0 \Longrightarrow \text{z} = \text{w}.$ Set w = 1. $\therefore \ z = 1.$ Put in (1) and (2).

$\therefore \ (1) \Longrightarrow y + 2 = 2x.$ And $(2) \Longrightarrow x + 2 = 2y.$

$\therefore \ y = 2x - 2.$ and $x + 2 = 2(2x - 2).$

$\therefore \ y = 2(2) - 2$ and $3x = 6.$ $\therefore \ y = 2$ and $x = 2.$

$\therefore \ u = (2, 2, 1, 1)$ is the fixed vector. Now 2+2+1+1 = 6. $\therefore$ Divide this vector u by 6.

$\therefore \ t = \left(\frac{2}{6}, \frac{2}{6}, \frac{1}{6}, \frac{1}{6}\right) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$ is the unique fixed probability vector of P.

$\therefore$ In long run, each boy receives the ball $\frac{1}{3}$ of the time and each girl $\frac{1}{6}$ of the time.

9. A man's smoking habits are as follows. If he smokes filter cigarettes one week, he switches to non-filter cigarettes the next week with probability 0.2. On the other hand, if he smokes non filter cigarettes one week there is a probability of 0.7 that he will smoke non filter cigarettes the next week as well. In the long run how often does he smoke filter cigarettes?

Solution:

The state space is {a(filter), b(non-filter)}. The transition matrix is $P = \begin{matrix} & a & b \\ a & 0.8 & 0.2 \\ b & 0.3 & 0.7 \end{matrix}$

Let t = (x, $1 - x$) be the fixed point of P, then t P = t.

$$\therefore \quad (x, 1 - x)\begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix} = (x, 1 - x)$$

$\therefore \quad 0.8x + 0.3 - 0.3x = x$ and $0.2x + 0.7 - 0.7x = 1 - x.$

$\therefore \quad 0.5x = 0.3. \quad \therefore \ x = \frac{0.3}{0.5} = \frac{3}{5} = 0.6. \quad \therefore \ t = (0.6, 1 - 0.6) = (0.6, \ 0.4).$

$\therefore$ In the long run, he smokes filter cigarettes 60% of the time.

10. A gambler's luck follows a pattern. If he wins a game, the probability of winning the next game is 0.6. However if he loses a game, the probability of losing the nest game is 0.7. There is an even chance of gambler winning the first game. If so (i) What is the probability of he winning the second game? (ii) What is the probability of he winning the third game? (iii) In the long run, how often he will win?

Solution:

The state space is {W(wins), L(lose)}. The transition matrix is $P = \begin{matrix} & W & L \\ W & 0.8 & 0.2 \\ L & 0.3 & 0.7 \end{matrix}$.

$\therefore \quad P = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{3}{10} & \frac{7}{10} \end{pmatrix}.$

$p^{(0)} = \left(\frac{1}{2}, \frac{1}{2}\right) = (0.5,\ 0.5)$. [∴ even chance of win].

(i) $p^{(1)} = p^{(0)}P = \left(\frac{1}{2}, \frac{1}{2}\right)\begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{3}{10} & \frac{7}{10} \end{pmatrix} = \left(\frac{9}{20}, \frac{11}{20}\right).$

∴ Probability of winning the second game is $\frac{9}{20}$.

(ii) $p^{(2)} = p^{(1)}P = \left(\frac{9}{20}, \frac{11}{20}\right)\begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{3}{10} & \frac{7}{10} \end{pmatrix} = \left(\frac{87}{200}, \frac{113}{200}\right)$

∴ Probability that he wins the third game is $\frac{87}{200}$.

(iii) Let t = (x, $1 - x$) be the unique fixed probability vector then t P = t.

$\therefore \quad (x, 1-x)\begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{3}{10} & \frac{7}{10} \end{pmatrix} = (x, 1-x).$

$\therefore \quad \frac{3x}{5} + \frac{3}{10} - \frac{3x}{10} = x$ and $\frac{2x}{5} + \frac{7}{10} - \frac{7x}{10} = 1 - x.$

$\therefore \quad \frac{3}{10} = \frac{7x}{10}. \quad \therefore \quad x = \frac{3}{7}. \quad \therefore \quad t = \left(\frac{3}{7}, 1 - \frac{3}{7}\right) = \left(\frac{3}{7}, \frac{4}{7}\right).$

∴ In the long run, he wins $\frac{3}{7}$ of the time.

11. Each year a man trades his car for a new car. If he has a Rover he trades it for a Vauxhall. If he has a Vauxhall, he trades it for a Ford. However, if he has a Ford, he is just as likely to trade it for a new Ford as to trade it for a Rover or a Vauxhall. In 2015

he brought his first car which was a Ford. (i) Find the probability that he has a (i) 2017 Ford, (ii) 2017 Rover, (iii) 2018 Vauxhall, (iv) 2018 Ford, (v) In the long run, how often will he have a Ford?

Solution:

The state space is {R(Rover), V(Vauxhall), F(Ford)}. The transition matrix of the Markov chain

$$\text{is } P = \begin{matrix} \\ R \\ V \\ F \end{matrix}\begin{matrix} \begin{matrix} R & V & F \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{matrix}$$

The initial probability vector is $p^{(0)} = (0,\ 0, 1)$. [Year 2015]

$$\therefore\ p^{(1)} = p^{(0)}P = (0,\ 0,\ 1)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \left(\frac{1}{3},\ \frac{1}{3}, \frac{1}{3}\right).\ \text{[Year 2016]}$$

$$\therefore\ p^{(2)} = p^{(1)}P = \left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \left(\frac{1}{9},\ \frac{4}{9},\ \frac{4}{9}\right).\ \text{[Year 2017]}$$

$$\therefore\ p^{(3)} = p^{(2)}P = \left(\frac{1}{9},\ \frac{4}{9},\ \frac{4}{9}\right)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \left(\frac{4}{27}, \frac{7}{27},\ \frac{16}{27}\right).\ \text{[Year 2018]}$$

∴ Probability that he has a

(i) 2017 Ford is $\frac{4}{9}$. (ii) 2017 Rover is $\frac{1}{9}$. (iii) 2018 Vauxhall is $\frac{7}{27}$. (iv) 2018 Ford is $\frac{16}{27}$.

Let t = (x, y, $1 - x - y$) be the unique fixed probability vector of P, then t P = t.

$$\therefore\ (x, y, 1 - x - y)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = (x, y, 1 - x - y)$$

$$\therefore\ \frac{1}{3} - \frac{x}{3} - \frac{y}{3} = x, \quad x + \frac{1}{3} - \frac{x}{3} - \frac{y}{3} = y, \quad y + \frac{1}{3} - \frac{x}{3} - \frac{y}{3} = 1 - x - y$$

$$\therefore\ 1 - x - y = 3x, \quad 3x + 1 - x - y = 3y, \quad y = 3 - 3x - 3y$$

$$\therefore\ y = 1 - 4x, \quad 2x - 4y = -1. \quad \therefore\ 2x - 4(1 - 4x) = -1. \quad \therefore\ 18x = 3$$

$$\therefore\ x = \frac{1}{6}. \quad \therefore\ y = 1 - 4\left(\frac{1}{6}\right). \quad \therefore\ y = \frac{1}{3}.$$

$\therefore\ t = \left(\frac{1}{6}, \frac{1}{3}, 1 - \frac{1}{6} - \frac{1}{3}\right) = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right).$

$\therefore$ In the long run he have Ford car $\frac{1}{2}$ of the time. i.e., 50% of the time.

12. For a Markov chain, the transition matrix is $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$ with initial probability distribution $p^{(0)} = \left(\frac{1}{4}, \frac{3}{4}\right)$. Find (i) $p_{21}^{(2)}$, (ii) $p_{12}^{(2)}$, (iii) $p^{(2)}$, (iv) $p_1^{(2)}$, (v) the vector $p^{(0)}P^n$ approaches, (vi) the matrix P^n approaches.

Solution:

Given $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$.

$$\therefore\ P^2 = P.P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{9}{16} & \frac{7}{16} \end{pmatrix}.$$

(i) $p_{21}^{(2)} = \frac{9}{16}$ and (ii) $p_{12}^{(2)} = \frac{3}{8}$.

$$\text{(iii)}\ p^{(2)} = p^{(0)}P^2 = \left(\frac{1}{4}, \frac{3}{4}\right)\begin{pmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{9}{16} & \frac{7}{16} \end{pmatrix} = \left(\frac{37}{64}, \frac{27}{64}\right).$$

(iv) $p_1^{(2)} = \frac{37}{64}$.

(v) $p^{(0)}P^n$ approaches the unique fixed probability vector t of P

Let t = (x, $1 - x$) be the fixed point of P then t P = t.

$$\therefore (x, 1 - x)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = (x, 1 - x).$$

$\therefore\ \frac{x}{2} + \frac{3}{4} - \frac{3x}{4} = x$ and $\frac{x}{2} + \frac{1}{4} - \frac{x}{4} = 1 - x.$

$\therefore\ 2x + 3 - 3x = 4x.\quad \therefore\ 5x = 3.\quad \therefore x = \frac{3}{5} = 0.6.$

$\therefore\ t = (0.6, 1 - 0.6) = (0.6,\ 0.4).$

$\therefore\ p^{(0)}P^n$ approaches the unique fixed probability vector $t = (0.6,\ 0.4)$.

(vi) P^n approaches the matrix T whose rows are each the fixed probability vector of P.

$\therefore$ P^n approaches $\begin{pmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{pmatrix}$

13. For a Markov chain, the transition matrix is $P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$ with initial probability distribution $p^{(0)} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$. Find (i) $p_{13}^{(2)}$, (ii) $p_{23}^{(2)}$, (iii) $p^{(2)}$, (iv) $p_1^{(2)}$.

Solution:

Given $P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$.

$$P^2 = P.P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{11}{16} & \frac{1}{8} & \frac{3}{16} \end{pmatrix}.$$

(i) $p_{13}^{(2)} = \frac{3}{8}$ (ii) $p_{23}^{(2)} = \frac{1}{2}$.

(iii) $p^{(2)} = p^{(0)}P^2 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)\begin{pmatrix} \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{11}{16} & \frac{1}{8} & \frac{3}{16} \end{pmatrix} = \left(\frac{7}{16}, \frac{1}{8}, \frac{7}{16}\right)$.

(iv) $p_1^{(2)} = \frac{7}{16}$.

HOME WORK:

1. The transition probability matrix of a Markov chain is given by $P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}$ and the initial probability distribution is $p^{(0)} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$. Find (i) $p_{13}^{(2)}$ *and* $p_{23}^{(2)}$, (ii) $p^{(3)}$ *and* $p_1^{(3)}$, (iii) the vector that $p^{(0)}P^n$ approaches, (iv) the matrix that P^n approaches.

2. The transition probability matrix of a Markov chain is given by P $= \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and initial

probability distribution is $p^{(0)} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix}$ find the following $p_{21}^{(2)}$, $p_{12}^{(2)}$, $p^{(4)}$ *and* $p_1^{(4)}$.

3. A company executive changes his car every year. If he has a car of make A he changes over to a of make B. If he has a car of make B he changes over to a car of make C. If he has a car of make C, he is just as likely to change over to a car make C,B or A. If he had a car of make C in the year 2016 . Find the probability that he will have a car of (i) make A in 2018 (ii) make C in 2018 (iii) make B in 2018 and (iv) make C in 2019. In long run, how often will he have a car of make C?

JOINT PROBABILITY DISTRIBUTION:

Joint Probability distributions of two discrete random variables:

Let X and Y be two discrete random variables on a sample space S such that $X = \{x_1, x_2, \dots \dots . x_n\}$ and $Y = \{y_1, y_2, \dots \dots y_n\}$ and

$X \times Y = \{(x_1, y_1), (x_1, y_2), \dots . . (x_1, y_m), (x_2, y_1) \dots . . (x_n, y_m)\}$.Then, the function h on $X \times Y$ defined by $h(x_i, y_j) = P(X = x_i, Y = y_j)$ is called the joint distribution or joint probability function of X and Y and is usually given in the form of a table.

X \ Y	y_1	y_2	…………..	y_m	Sum
x_1	$h(x_1, y_1)$	$h(x_1, y_2)$	…………..	$h(x_1, y_m)$	$f(x_1)$
x_2	$h(x_2, y_1)$	$h(x_2, y_2)$	…………..	$h(x_2, y_m)$	$f(x_2)$
. . . .	. . . .	. . . .	…………..	. . . .	. . . .
x_n	$h(x_n, y_1)$	$h(x_n, y_2)$	…………..	$h(x_n, y_m)$	$f(x_n)$
Sum	$g(y_1)$	$g(y_2)$	…………..	$g(y_m)$	

Here $h(x_i, y_j)$ is the probability of the ordered pair (x_i, y_j).

The joint distribution h satisfies the conditions (i) $h(x_i, y_j) \geq 0$ and (ii) $\sum_{i=1}^{n} \sum_{j=1}^{m} h(x_i, y_j) = 1$.

Marginal distributions:

The function $f(x_i)$ and $g(y_i)$ are defined by $f(x_i) = \sum_{j=1}^{m} h(x_i, y_j)$ and $g(y_j) = \sum_{i=1}^{n} h(x_i, y_j)$ are called the marginal distributions and are the individual distributions of X and Y respectively.

Constants of joint distribution:

Expectations:

If X and Y are two discrete random variables with the joint distribution $h(x_i, y_j)$, then the expectations of X and Y are the means of X and defined by by $\mu_{\mathrm{X}} = E[\mathrm{X}] = \sum_{i=1}^{n} x_i\, f(x_i)$ and $\mu_{\mathrm{Y}} = E[\mathrm{Y}] = \sum_{j=1}^{m} y_j\, g(y_j)$.

Variance:

Variance of X and Y are defined by

$\sigma_{\mathrm{X}}^2 = var(\mathrm{X}) = \sum_{i=1}^{n} (x_i - \mu_{\mathrm{X}})^2\, f(x_i) = \sum_{i=1}^{n} x_i^2\, f(x_i) - \mu_{\mathrm{X}}^2 = E[\mathrm{X}^2] - \{E[\mathrm{X}]\}^2$

where $E[\mathrm{X}^2] = \sum_{i=1}^{n} x_i^2\, f(x_i)$ and

$\sigma_{\mathrm{Y}}^2 = var(\mathrm{Y}) = \sum_{j=1}^{m} \left(y_j - \mu_{\mathrm{Y}}\right)^2 g(y_j) = \sum_{j=1}^{m} y_j^2\, g(y_j) - \mu_{\mathrm{Y}}^2 = E[\mathrm{Y}^2] - \{E[\mathrm{Y}]\}^2$

where $E[\mathrm{Y}^2] = \sum_{j=1}^{m} y_j^2\, g(y_j)$.

Standard deviation:

Standard deviation of X and Y are defined by $\sigma_{\mathrm{X}} = \sqrt{var(\mathrm{X})}$ and $\sigma_{\mathrm{Y}} = \sqrt{var(\mathrm{Y})}$.

Covariance:

Covariance of X and Y is denoted by $Cov(X, Y)$ and is defined by

$Cov(X, Y) = \sum_{i,j} (x_i - \mu_x)(y_j - \mu_y)h(x_i, y_j) = E[(X - \mu_x)(Y - \mu_y)]$ (or)

$Cov(X, Y) = \sum_{i,j} x_i y_j h(x_i, y_j) - \mu_x \mu_y = E[X, Y] - \mu_x \mu_y$, where $E[X, Y] = \sum_{i,j} x_i y_j h(x_i, y_j)$

Correlation:

Correlation of X and Y is denoted by $\rho(X, Y)$ and is defined by $\rho(X, Y) = \frac{Cov(X,Y)}{\sigma_X\, \sigma_Y}$.

Independent Random Variables:

Two random variables X and Y are said to be independent if

$P(X = x_i, Y = y_j) = P(X = x_i).P(Y = y_j)$ i.e. $h(x_i, y_j) = f(x_i)g(y_j)$.

Problems:

1. Let X and Y be two random variables with the following joint distribution

Find (i) the distributions of X and Y (i.e. Marginal distributions), (ii) Cov(X, Y), (iii) $\rho(X, Y)$. Also, (iv) Show that the random variables X and Y are independent.

Solution:

(i) Given

X \ Y	2	3	4	Sum
1	0.06	0.15	0.09	0.30
2	0.14	0.35	0.21	0.70
Sum	0.20	0.50	0.30	

∴ Marginal distributions of X and Y are as follows

Distributionof X

$X = x_i$	1	2
$f(x_i)$	0.30	0.70

Distribution of Y

$Y = y_j$	2	3	4
$g(y_j)$	0.20	0.50	0.30

(ii) Mean $\mu_x = E[X] = \sum x_i f(x_i)$ = 1(0.3) + 2(0.7) = 1.7 and

$\mu_Y = E[Y] = \sum y_j g(y_j)$ = 2(0.2) + 3(0.5) + 4(0.3) = 3.1

$E[XY] = \sum_{i,j} x_i\, y_j h(x_i, y_j)$

= (1)(2)(0.06) + (1)(3)(0.15) + (1)(4)(0.09) + (2)(2)(0.14) + (2)(3)(0.35) + (2)(4)(0.21)

$E[XY]$ = 5.27

$Cov(X,Y) = E[XY] - \mu_x\mu_Y = 5.27 - 1.7(3.1) = 0.$

(iii) $\sigma_x{}^2 = var(X) = \sum x_i{}^2 f(x_i) - \mu_x{}^2 = 1^2(0.3) + 2^2(0.7) - (1.7)^2 = 0.21.$

∴ $\sigma_x = 0.4582.$

$\sigma_y{}^2 = var(Y) \sum y_j{}^2 g(y_j) - \mu_y{}^2 = 2^2(0.2) + 3^2(0.5) + 4^2(0.3) - (3.1)^2 = 0.49.$ ∴ $\sigma_y = 0.7$

$$\therefore \rho(X,Y) = \frac{Cov(X,Y)}{\sigma_x\sigma_y} = 0$$

(iv) From the table we observe that $h(x_i, y_j) = f(x_i)g(y_j)$.

i.e. h(1,2) = 0.06 and f(1)g(2) = (0.3)(0.2) = 0.06 ∴ h(1,2) = f(1)g(2), etc.

∴ X and Y are independent.

NOTE:

If X and Y are independent random variables then

(i) E[XY] = E(X)E(Y), (ii) Var(X+Y) = Var(X) + Var(Y), (iii) Cov(X,Y) = 0.

2. Suppose X and Y have the following joint distribution

X \ Y	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

Find (i) the distributions of X and Y, (ii) Cov(X, Y), (iii) $\rho(X, Y)$, (iv) Are X and Y independent random variables?

Solution:

(i) Given

X \ Y	-3	2	4	sum
1	0.1	0.2	0.2	0.5
3	0.3	0.1	0.1	0.5
Sum	0.4	0.3	0.3	

∴ Marginal distributions of X and Y are as follows

Distribution of X

$X = x_i$	1	3
$f(x_i)$	0.5	0.5

Distribution of Y

$Y = y_j$	-3	2	4
$g(y_j)$	1	0.3	0.3

ii] Mean $\mu_x = E[X] = \sum x_i f(x_i) = 1(0.5) + 3(0.5) = 2$

$\mu_Y = E[Y] = \sum y_j g(y_j) = (-3)(1) + 2(0.3) + 4(0.3) = 0.6$

$E[XY] = \sum_{i,j} x_i y_j h(x_i, y_j)$

$= 1(-3)(0.1) + 1(2)(0.2) + 1(4)(0.2) + 3(-3)(0.3) + 3(2)(0.1) + 3(4)(0.1)$

$E[XY] = 0.$

$\therefore Cov(X,Y) = E[XY] - \mu_x \mu_Y = 0 - 1.2 = -1.2$

(iii) ${\sigma_x}^2 = \sum {x_i}^2 f(x_i) - {\mu_x}^2 = 1^2(0.5) + 3^2(0.5) - 2^2 = 1.$ $\therefore \sigma_x = 1.$

${\sigma_y}^2 = \sum {y_j}^2 g(y_j) - {\mu_y}^2 = (-3)^2(1) + 2^2(0.3) + 4^2(0.3) - (0.6)^2 = 9.24.$ $\therefore \sigma_y = 3.0397$

$\therefore \rho(X,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = \frac{(-1.2)}{(1)(3.0397)} = -0.39 \approx -0.4.$

(iv) We observe that $h(x_i, y_j) = f(x_i)g(y_j)$ i.e. h(1,-3) = 0.1 and f(1)g(-3) = (0.5)(0.4) = 0.2

∴ h(1,-3) ≠ f(1)g(-3)

∴ X and Y are not independent.

3. Consider the following joint distribution of X and Y:

X \ Y	-4	2	7	sum
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$
5	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{2}$
sum	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{4}$	

Find (i) E(X) and E(Y), (ii) Cov(X,Y), (iii) σ_x , σ_y and $\rho(X, Y)$.

Solution:

From the given table Marginal distributions of X and Y are as follows:

Distribution of X

$X = x_i$	1	5
$f(x_i)$	$\frac{1}{2}$	$\frac{1}{2}$

Distribution of Y

$Y = y_j$	-4	2	7
$g(y_j)$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{4}$

(i) Mean $\mu_x = E[X] = \sum x_i f(x_i) = 1\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right) = 3$

$\mu_Y = E[Y] = \sum y_j g(y_j) = (-4)\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 7\left(\frac{1}{4}\right) = 1$

(ii) $E[XY] = \sum_{i,j} x_i\, y_j h(x_i, y_j)$

$$= 1(-4)\left(\frac{1}{8}\right) + 1(2)\left(\frac{1}{4}\right) + 1(7)\left(\frac{1}{8}\right) + 5(-4)\left(\frac{1}{4}\right) + 5(2)\left(\frac{1}{8}\right) + 5(7)\left(\frac{1}{8}\right) = \frac{3}{2}$$

$Cov(X, Y) = E[XY] - \mu_x \mu_Y = \frac{3}{2} - 3(1) = -1.5$

(iii) $\sigma_x{}^2 = \sum x_i{}^2 f(x_i) - \mu_x{}^2 = 1^2\left(\frac{1}{2}\right) + 5^2\left(\frac{1}{2}\right) - 3^2 = 4.$ $\therefore \sigma_x = 2.$

$\sigma_y{}^2 = \sum y_j{}^2 g(y_j) - \mu_y{}^2 = (-4)^2\left(\frac{3}{8}\right) + 2^2\left(\frac{3}{8}\right) + 7^2\left(\frac{1}{4}\right) - (1)^2 = \frac{75}{4}.$ $\therefore \sigma_y = 4.33$

$$\therefore \rho(X,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = \frac{(-1.5)}{(2)(4.33)} = -0.173$$

4. Consider the following joint distribution of X and Y:

X \ Y	-2	-1	4	5	sum
1	0.1	0.2	0	0.3	0.6
2	0.2	0.1	0.1	0	0.4
sum	0.3	0.3	0.1	0.3	

Find (i) E(X) and E(Y), (ii) Cov(X,Y), (iii) σ_x , σ_y and $\rho(X,Y)$

Soln: From the given table Marginal distributions of X and Y are as follows:

Distribution of X

x_i	1	2
$f(x_i)$	0.6	0.4

Distribution of Y

y_j	-2	-1	4	5
$g(y_j)$	0.3	0.3	0.1	0.3

(i) Mean $\mu_x = E[X] = \sum x_i f(x_i) = 1(0.6) + 2(0.4) = 1.4$

$$\mu_Y = E[Y] = \sum y_j g(y_j) = (-2)(0.3) + (-1)(0.3) + 4(0.1) + 5(0.3) = 1$$

(ii) $E[XY] = \sum_{i,j} x_i\, y_j h(x_i, y_j)$

$$= 1(-2)(0.1) + 1(-1)(0.2) + 1(4)(0) + 1(5)(0.3) + 2(-2)(0.2) + 2(-1)(0.1)$$
$$+2(4)(0.1) + 2(5)(0).$$

$E[XY] = 0.9.$

$Cov(X,Y) = E[XY] - \mu_x \mu_Y = (0.9) - (1.4)(1) = -0.5$

(iii) ${\sigma_x}^2 = \sum {x_i}^2 f(x_i) - {\mu_x}^2 = 1^2(0.6) + 2^2(0.4) - (1.4)^2 = 0.24.$ $\therefore \sigma_x = 0.49.$

${\sigma_y}^2 = \sum {y_j}^2 g(y_j) - {\mu_y}^2 = (-2)^2(0.3) + (-1)^2(0.3) + 4^2(0.1) + 5^2(0.3) - (1)^2 = 9.6.$

$\therefore \sigma_y = 3.09$

$$\therefore \rho(X,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = \frac{(-0.5)}{(0.49)(3.09)} = -0.33$$

5. Let X and Y are independent variables with the following distributions:

Distribution of X

x_i	1	2
$f(x_i)$	0.6	0.4

Distribution of Y

y_j	5	10	15
$g(y_j)$	0.2	0.5	0.3

Find the joint distribution h of X and Y.

Solution:

Given Marginal distributions of X and Y are:

Distribution of X

x_i	1	2
$f(x_i)$	0.6	0.4

Distribution of Y

y_j	5	10	15
$g(y_j)$	0.2	0.5	0.3

Since X and Y are independent, we have $h(x_i, y_j) = f(x_i)g(y_j)$

∴ h(1,5) = f(1)g(5) = (0.6)(0.2) = 0.12

h(1,10) = f(1)g(10) = (0.6)(0.5) = 0.30

h(1,15) = f(1)g(15) = (0.6)(0.3) = 0.18

h(2,5) = f(2)g(5) = (0.4)(0.2) = 0.80

h(2,10) = f(2)g(10) = (0.4)(0.5) = 0.20

h(2,15) = f(2)g(15) = (0.4)(0.3) = 0.12

∴ The joint distribution h of X and Y is

Y / X	5	10	15	sum
1	0.12	0.30	0.18	0.6
2	0.08	0.20	0.12	0.4
Sum	0.20	0.50	0.30	

6. Suppose X and Y are independent random variables with the following respective distributions

Find the joint distribution of X and Y, and verify that Cov(X,Y) = 0.

x_i	1	2
$f(x_i)$	0.7	0.3

y_j	-2	5	8
$g(y_j)$	0.3	0.5	0.2

Solution:

Given Marginal distributions of X and Y are:

Distribution of X

x_i	1	2
$f(x_i)$	0.7	0.3

Distribution of Y

y_j	-2	5	8
$g(y_j)$	0.3	0.5	0.2

Since X and Y are independent, we have $h(x_i, y_j) = f(x_i)g(y_j)$

∴ h(1,-2) = f(1)g(-2) = (0.7)(0.3) = 0.21

h(1,5) = f(1)g(5) = (0.7)(0.5) = 0.35

h(1,8) = f(1)g(8) = (0.7)(0.2) = 0.14

h(2,-2) = f(2)g(-2) = (0.3)(0.3) = 0.09

h(2,5) = f(2)g(5) = (0.3)(0.5) = 0.15

h(2,8) = f(2)g(8) = (0.3)(0.2) = 0.06

∴ The joint distribution of X and Y

X \ Y	-2	5	8	sum
1	0.21	0.35	0.14	0.7
2	0.09	0.15	0.06	0.3
Sum	0.3	0.5	0.2	

Mean $\mu_x = E[X] = \sum x_i f(x_i) = 1(0.7) + 2(0.3) = 1.3$

$\mu_Y = E[Y] = \sum y_j g(y_j) = (-2)(0.3) + 5(0.5) + 8(0.2) = 3.5$

$E[XY] = \sum_{i,j} x_i\, y_j h(x_i, y_j)$

$= 1(-2)(0.21) + 1(5)(0.35) + 1(8)(0.14) + 2(-2)(0.09) + 2(5)(0.15) + 2(8)(0.06)$

∴ $E[XY] = 4.55$

∴ $Cov(X, Y) = E[XY] - \mu_x\mu_Y = (4.55) - (1.3)(3.5) = 0.$

7. Let X be a random variable with the following distribution and Y is defined to be X^2:

x_i	-2	-1	1	2
$f(x_i)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Determine (i) the distribution g of Y, (ii) the joint distribution h of X and Y, (iii) Cov (X,Y) and $\rho(X,Y)$.

Solution:

(i) Given

x_i	-2	-1	1	2
$f(x_i)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Let $Y = X^2$, then Y = 4 when X = -2 or 2 and Y = 1 when X = -1 or 1.

∴ The random variable Y the values 1 and 4.

$\therefore g(1) = P(Y = 1) = P(X = -1 \text{ or } X = 1) = P(X = -1) + P(X = 1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

$g(4) = P(Y = 4) = P(X = -2 \text{ } or \text{ } X = 2) = P(X = -2) + P(X = 2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

∴ The distribution of g on Y is

y_j	1	4
$g(y_j)$	$\frac{1}{2}$	$\frac{1}{2}$

(ii) The joint distribution h of X and Y is as follows:

Now, S = {(-2,4), (-1,1), (1,1), (2,4)} and $h(x_i, y_j) = P(X = x_i, Y = y_j)$.

when $X = -2, Y = X^2 = 4$.

$\therefore h(-2,1) = P(X = -2, Y = 1) = 0.$ ($\because$ $when\ X = -2, Y \neq 1$).

and $h(-2,4) = P(X = -2, Y = 4) = \frac{1}{4}$.

when $X = -1, Y = X^2 = 1$.

$\therefore h(-1,1) = P(X = -1, Y = 1) = \frac{1}{4}$ and $h(-1,4) = P(X = -1, Y = 4) = 0$.

when $X = 1, Y = X^2 = 1$.

$\therefore h(1,1) = P(X = 1, Y = 1) = \frac{1}{4}$ and $h(1,4) = P(X = 1, Y = 4) = 0$.

when $X = 2, Y = X^2 = 4$.

$\therefore h(2,1) = P(X = 2, Y = 1) = 0$ and $h(2,4) = P(X = 2, Y = 4) = \frac{1}{4}$.

Y \ X	1	4	sum
-2	0	$\frac{1}{4}$	$\frac{1}{4}$
-1	$\frac{1}{4}$	0	$\frac{1}{4}$
1	$\frac{1}{4}$	0	$\frac{1}{4}$
2	0	$\frac{1}{4}$	$\frac{1}{4}$
sum	$\frac{1}{2}$	$\frac{1}{2}$	

(iii) $\mu_x = E[X] = \sum x_i f(x_i) = (-2)\left(\frac{1}{4}\right) + (-1)\left(\frac{1}{4}\right) + 1\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right) = 0.$

$\mu_Y = E[Y] = \sum y_j g(y_j) = 1\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) = \frac{5}{2}.$

$E[XY] = \sum_{i,j} x_i\, y_j h(x_i, y_j) = 0 - 2 - \frac{1}{4} + 0 + \frac{1}{4} + 0 + 0 + 2 = 0.$

$\therefore Cov(X, Y) = E[XY] - \mu_x \mu_Y = 0 - (0)\left(\frac{5}{2}\right) = 0.$

$\therefore \rho(X, Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = 0.$

8. A fair coin is tossed three times. Let X denote 0 and 1 according as a head or tail occurs on the first toss, and let Y denote the number of heads which occur. Determine (i) the distribution of X and Y, (ii) the joint distribution h of X and Y, (iii) Cov (X, Y).

Solution:

(i) The sample space S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

Let X = {0, 1} where 0 for head and 1 for tail on the first toss.

Let Y = number of heads i.e. Y = {0,1,2,3}.

$\therefore f(0) = P(X = 0) = \frac{1}{2}$, $f(1) = P(X = 1) = \frac{1}{2}$

$g(0) = P(Y = 0) = \frac{1}{8}$, $g(1) = P(Y = 1) = \frac{3}{8}$,

$g(2) = P(Y = 2) = \frac{3}{8}$, $g(3) = P(Y = 3) = \frac{1}{8}$

∴ The distributions of X and Y are:

Distribution of X

x_i	0	1
$f(x_i)$	$\frac{1}{2}$	$\frac{1}{2}$

Distribution of Y

y_j	0	1	2	3
$g(y_j)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(ii) The joint distribution h of X and Y is given by $h(x_i, y_j) = P(X = x_i, Y = y_j)$.

h(0, 0) = P(X=0, Y=0) = 0

h(0, 1) = P(X=0, Y=1) = $\frac{1}{8}$

h(0, 2) = P(X=0, Y=2) = $\frac{1}{4}$

h(0, 3) = P(X=0, Y=3) = $\frac{1}{8}$

h(1, 0) = P(X=1, Y=0) = $\frac{1}{8}$

h(1, 1) = P(X=1, Y=1) = $\frac{1}{4}$

h(1, 2) = P(X=1, Y=2) = $\frac{1}{8}$

h(1, 3) = P(X=1, Y=3) = 0

X \ Y	0	1	2	3	sum
0	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{1}{2}$
sum	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	

(iii) $\mu_x = E[X] = \sum x_i f(x_i) = 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = \frac{1}{2}$

$\mu_Y = E[Y] = \sum y_j g(y_j) = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) = \frac{3}{2}$

$E[XY] = \sum_{i,j} x_i\, y_j h(x_i, y_j) = 0 + 0 + 0 + 0 + 0 + 1(1)\left(\frac{1}{4}\right) + 1(2)\left(\frac{1}{8}\right) + 0 = \frac{1}{2}$

$Cov(X,Y) = E[XY] - \mu_x\mu_Y = \frac{1}{2} - \frac{1}{2}\left(\frac{3}{2}\right) = -\frac{1}{4}.$

9. Determine i) Marginal distribution, ii) Covariance between the discrete random variables X and Y, using the joint probability distribution:

X \ Y	3	4	5	sum
2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
5	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
7	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
sum	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

Solution:

(i) Given Marginal distributions of X and Y are:

Distribution of X

x_i	2	5	7
$f(x_i)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Distribution of Y

y_j	3	4	5
$g(y_j)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

(ii) Mean $\mu_x = E[X] = \sum x_i f(x_i) = 2\left(\frac{1}{2}\right) + 5\left(\frac{1}{4}\right) + 7\left(\frac{1}{4}\right) = 4$

$\mu_Y = E[Y] = \sum y_j g(y_j) = 3\left(\frac{1}{3}\right) + 4\left(\frac{1}{3}\right) + 5\left(\frac{1}{3}\right) = 4.$

$E[XY] = \sum_{i,j} x_i\, y_j h(x_i, y_j)$

$= (2)(3)\left(\frac{1}{6}\right) + (2)(4)\left(\frac{1}{6}\right) + (2)(5)\left(\frac{1}{6}\right) + (5)(3)\left(\frac{1}{12}\right) + (5)(4)\left(\frac{1}{12}\right) + (5)(5)\left(\frac{1}{12}\right) + (7)(3)\left(\frac{1}{12}\right) + (7)(4)\left(\frac{1}{12}\right) + (7)(5)\left(\frac{1}{12}\right)$

$E[XY] = 16.$

$Cov(X,Y) = E[XY] - \mu_x\mu_Y = 16 - 4(4) = 0.$

HOME WORK:

1. The Joint distribution of two random variables X and Y is as follows

X \ Y	0	1	2	3
0	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0

Compute the following (i) E(X) and E(Y), (ii) E(XY), (iii) σ_x and σ_y, (iv) cov(X,Y), (v) $\rho(X,Y)$.

2. The Joint Probability distribution of two random variables X and Y is given by the following table:

X \ Y	-2	5	8
1	0.21	0.35	0.14
2	0.09	0.15	0.06

Determine the Marginal distribution of X and Y. Also find (i) E(X) and E(Y), (ii) E(XY), (iii) Standard deviation of X and Y, (iv) Covariance of X and Y, (v) Correlation of X and Y.