Principles of Quantum Information & Quantum Computing

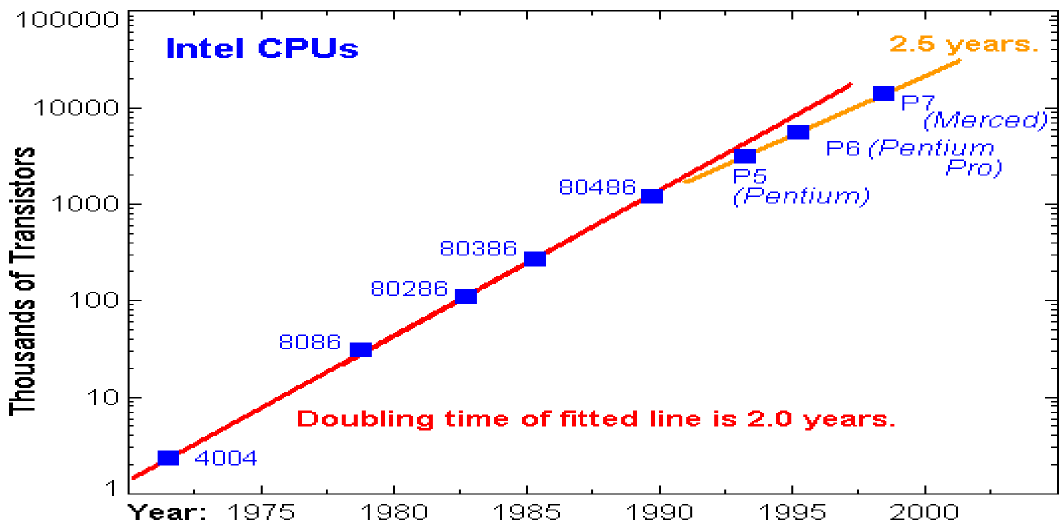
Introduction to Quantum Computing:

Quantum Computing is the area of study focused on developing computing methods based on the principle of quantum theory. Quantum computing is based on the principle of quantum superposition. In Quantum computing, the information is encoded in quantum system such as atoms, ions or quantum dots.

Moore's law & its end:

In the year 1965, Gordon Moore observed increasing performance in the first few generations of the integrated circuit (IC) technology. Moore predicted that it would continue to improve at an exponential rate with the performance per unit cost increasing by a factor or two every 18 months are so. The Demise of the Transistor in the quantum scale could be expected as the dimensions decrease further. The quantum computations are the option for future generation.

Statement–"It states that the number of transistors on a microchip doubles every two years."



Differences Between Classical and Quantum Computing:

Classical Computing	Quantum Computing
Used by large scale, multipurpose devices.	Used by high speed, quantum mechanics-based computers.
Information is stored in bits.	Information is based on Quantum bits.
There is discrete number of possible states either 0 or 1.	There is infinite, continuous number of possible states. They are the result of Quantum superposition.
Calculations are deterministic. This means repeating the same inputs results in the Same output.	The calculations are probabilistic, meaning there are multiple possible outputs to the Same inputs.
Data processing is carried out by logic and In sequential order.	Data processing is carried out by quantum logic at parallel instances.
Operations are governed by Boolean Algebra.	Operations are defined by linear algebra by Hilbert Space.

Circuit behavior is defined by Classical Physics.	Circuit behavior is defined by quantum Mechanics.
Very robust, immune to noise even at 400 K.	Very sensitive to noise, operates at close to absolute zero temperature.

Concept of Qubit and its properties:

The counterpart of a classical bit in quantum computing is Qubit. It's the basic unit of information in a quantum computer. Superposition, Entanglement, and Tunneling are all special properties that define a qubit.

Properties of Qubits

1. A qubit can be in a superposed state of the two states 0 and 1.

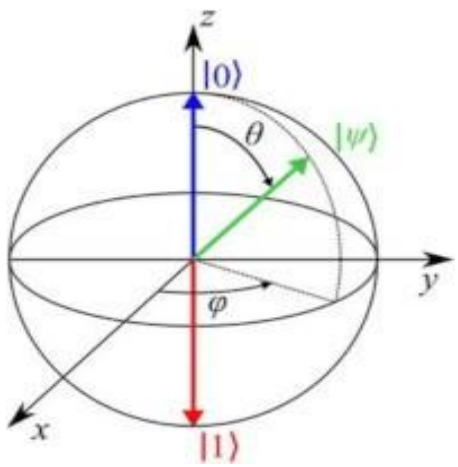
2. If measurements are carried out with a qubit in superposed state then the results that are probabilistic in nature.

3. Owing to the quantum nature, the qubit changes its state at once when subjected to measurement. This means, one cannot copy information from qubits the way we do in the present computers and is known as "no cloning principle".

4. A Qubit can be physically implemented by the two states of an electron or horizontal and vertical polarizations of photons as |↓> and |↑>.

Representation of Qubits by Bloch Sphere

The pure state space qubits (Two Level Quantum Mechanical Systems) can be visualized using an imaginary sphere called *Bloch Sphere*. It has a unit radius.



The Arrow on the sphere represents the state of the Qubit. The north and south poles are used to represent the basis states |0⟩ and l1> respectively. The other locations are the super positions of 0 and 1states and represented. Thus a qubit can be any point on the Bloch Sphere.

$$|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$$

Where α and β are the probability amplitudes satisfying the condition$|\alpha|^2+|\beta|^2=1$.

The Bloch sphere allows the state of the qubit to be represented by unit spherical co-ordinates. They are the polar angle θ and the azimuth angle ϕ. The Bloch sphere is represented by the equation,

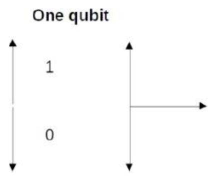
$$|\psi\rangle=\cos\frac{\theta}{2}|0\rangle+e^{i\pi}\sin\frac{\theta}{2}|1\rangle$$

Here $0 \le \theta \le \pi$ and Here $0 \le \varphi \le 2\pi$. The normalization constant is given by

$$\left|cos\frac{\theta}{2}\right|^2 + \left|sin\frac{\theta}{2}\right|^2 = 1$$

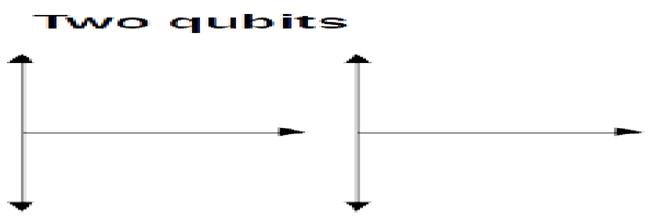
Single qubit

A Single qubit has two computational basis states $|0\rangle$ and$|1\rangle$. The pictorial representation of the single qubit is as follows: $\alpha|0\rangle + \beta|1\rangle$



Two qubit

A two-qubit system has 4 computational basis states denoted as $|00\rangle,|01\rangle,|10\rangle,|11\rangle$. The pictorial representation of the two qubit is as follows: $\alpha|0\rangle + \beta|1\rangle+ \gamma\ |10\rangle + \delta|11\rangle$

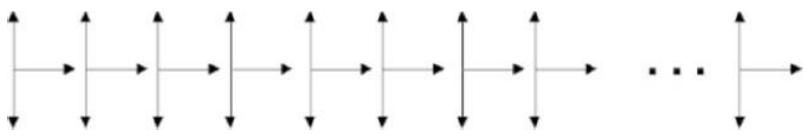


N qubits

A multi-qubit system of N qubits has 2^n computational basis states. For example, a state with 3 qubits has 2^3 computational basis states. Thus, for N qubits the computational basis states denoted as

$|00\cdots\cdots00\rangle,|00\cdots\cdots01\rangle,|00\cdots\cdots10\rangle,|00\cdots\cdots11\rangle\cdots\cdots\cdots\cdots|11\cdots\cdots11\rangle$.

The block diagram of representation of N qubits is as follows,



Dirac Representation and Matrix Operations:

Linear Algebra is the study of vector spaces and operations on vector spaces. The Standard quantum mechanical notation for a quantum state k in a vector space is $|k\rangle$. The notation $|\rangle$ indicates that the object is a vector r and is called a *ket vector*. The examples of ket vectors are $|i\rangle,|D\rangle$ etc.

Matrix Representation of 0 and 1

For qubits the computational basis (0 and 1) is represented by two ket vectors: $|0\rangle$=0 and $|1\rangle$=1. These vectors can be represented as column vectors,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The quantum state $|\psi\rangle$ in the equation $|\psi\rangle = \alpha|0\rangle+\beta|1\rangle$ can be written as a unit column vector in a two-dimensional complex plane spanned by the two basis states known as normal basis. Here the vectors $|0\rangle$and $|1\rangle$ are orthogonal, i.e., perpendicular to each other. a qubit with states$|0\rangle$and$|1\rangle$ is represented by the column vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

$$|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$$

$$|\psi\rangle=\alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\beta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle=\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Thus an arbitrary state can be represent by $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

Identity operator (I)

The identity operator I is an operator, which operating on a state vector, leaves the state unchanged.

i.e.

$$I|a\rangle=|a\rangle$$

$$I|0\rangle=|0\rangle$$

$$I|1\rangle=|1\rangle$$

The Identity operator in the matrix form is given by,

$$I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let us consider the operation of Identity operator on $|0\rangle$ and $|1\rangle$states,

$$I|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\therefore I|0\rangle=|0\rangle$$

$$I\,|1\rangle= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore I|1\rangle=|1\rangle$$

Thus, the operation of identity matrix (operator) on$|0\rangle$ and $|1\rangle$ leaves the states unchanged.

Pauli Matrices and representation:

The Pauli matrices are the set of three 2×2 matrices which are Hermitian and unitary. Pauli matrices are usually denoted by the Greek letter sigma (σ).

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli matrices operating on I|0⟩ and I|1⟩ states:

1. $\sigma_0|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

   $\sigma_0 =|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

2. $\sigma_x|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

   $\sigma_x =|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

3. $\sigma_y|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle$

   $\sigma_y|1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|0\rangle$

4. $\sigma_Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

   $\sigma_Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle$

Conjugate of a Matrix

Conjugate of a matrix is formed by replacing each element of the matrix by its complex conjugate.

$$A = \begin{pmatrix} i & 1 \\ 0 & 2-3i \end{pmatrix}$$

The conjugate of a matrix A is given by $A^* = \begin{pmatrix} -i & 1 \\ 0 & 2+3i \end{pmatrix}$

Transpose of a matrix:

The transpose of a matrix is formed by interchanging its rows and columns.

$$A=\begin{pmatrix} i & 1 \\ 0 & 2-3i \end{pmatrix}$$

Transpose of A is given by $A^T = \begin{pmatrix} i & 0 \\ 1 & 2-3i \end{pmatrix}$

The conjugate transpose of a matrix:

The conjugate transpose of a matrix interchanges the rows and columns each element, reflecting the elements across the main diagonal. The operation also negatives the imaginary part of the complex numbers.

$$A=\begin{pmatrix} i & 1 \\ 0 & 2-3i \end{pmatrix}$$

The conjugate transpose of A is given by

$$A^\dagger= \begin{pmatrix} -i & 0 \\ 1 & 2+3i \end{pmatrix}$$

Hermitian:

The matrix that is equal to its conjugate transpose is called Hermitian. Thus $A^\dagger = A$. then it is called Hermitian or self adjoint matrix.

$$A=\begin{pmatrix} 3 & 3+i \\ 3-i & 2 \end{pmatrix}$$

Conjugate of A = $A^* = \begin{pmatrix} 3 & 3-i \\ 3+i & 2 \end{pmatrix}$

The transpose of $A^* = (A^*)^T = \begin{pmatrix} 3 & 3+i \\ 3-i & 2 \end{pmatrix} = A$

Unitary matrix :

If a matrix U is said to be unitary, the product of the matrix and the conjugate transpose of a matrix is equal to the identity matrix

$U .U^\dagger = I$

$U.U^{-1} = I$

For example $U=\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \end{bmatrix}$

$$U^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \end{bmatrix}$$

$$U .U^\dagger =\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$U .U^\dagger = I$$

Probability, quantum superposition and Normalization

Let us consider a quantum state or qubit $|\psi\rangle$in the form of $|0\rangle$and$|1\rangle$ state then we have,

$$|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$$

The above equation represents the quantum superposition of states$|0\rangle$and $|1\rangle$.

$$|\psi\rangle= \alpha\binom{1}{0} + \beta\binom{0}{1} = \binom{\propto}{\beta}$$

Also we have, $\langle\psi|= [\alpha^*\beta^*]$

Let us consider the inner product $\langle\psi|\psi\rangle$ i.e.

$$\langle\psi|\psi\rangle=[\alpha^*\beta^*]\binom{\alpha}{\beta}$$

$$\langle\psi|\psi\rangle=\alpha^*\alpha+\beta^*\beta$$

But $\alpha^*\alpha+\beta^*\beta=|\alpha|^2+|\beta|^2$

$$\therefore\langle\psi|\psi\rangle=|\alpha|^2+|\beta|^2$$

This can also be written as

$$|\psi|^2=\psi\psi^*$$

The above equation represents the probability density.

As per the principle of normalization we have,

$$|\psi|^2=\psi\psi^*=1$$

Or $\langle\psi|\psi\rangle=|\alpha|^2+|\beta|^2=1$

This implies that a quantum state$|\psi\rangle$is normalized.

Orthogonality:

If two quantum states $|\psi\rangle$ and $|\varphi\rangle$ are said to be orthogonal, if their inner product is zero.

Mathematically it can be written as $\langle\psi|\varphi\rangle = 0$

Let us consider the inner product of $|0\rangle$and $|1\rangle$ is $\langle 0|1\rangle=0$

Orthonormality

If two quantum states$|\psi\rangle$and$|\varphi\rangle$are said to be orthonormal if,

- $|\psi\rangle$and$|\varphi\rangle$are normalized.
- $|\psi\rangle$and $|\varphi\rangle$are orthogonal to each other.

Column and row matrix:

The column vectors are called ket vectors denoted by $|\psi\rangle$ and are represented by column matrices.

The row vectors are called Bra vectors denoted by $\langle\varphi|$ and are represent by row matrices.

Consider a ket vector represented in the form of column matrix is $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

The row matrix represented by $|\psi\rangle = (\alpha^* \beta^*)$

Here $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (\alpha^* \beta^*)$

Bra vectors are complex conjugate of ket vectors and vice- verse.

Quantum gates:

In quantum computing a quantum logic gate is a basic quantum circuit operating on a small number of qubits. A qubit is useless unless it is used to carry out a quantum calculation. The quantum calculations are achieved by performing a series of fundamental operations, known as quantum logic gates. They are the building blocks of quantum circuits similar to the classical logic gates in conventional digital circuits. Unlike many classical logic gates, quantum logic gates are reversible. It is possible to perform quantum computing using only reversible gates.

Single qubit gates:

Quantum NOT gate(X)

In quantum computing the quantum NOT gate for qubits takes the state $|0\rangle$ to $|1\rangle$ and vice versa. It is analogous to the classical not gate.

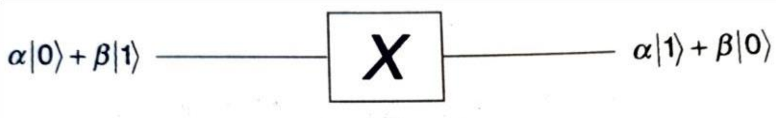
The matrix representation of quantum not gate and its operation on $|0\rangle$ and $|1\rangle$ are as follows is given by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

The quantum not gate represented by



Quantum NOT gate

Truth table of NOT gate

Input	Output
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\alpha\|0\rangle+\beta\|1\rangle$	$\alpha\|1\rangle+\beta\|0\rangle$

Pauli-X, Y and Z gates

The Pauli X gate is same as quantum Not gate

Pauli Y gate

The Y gate is represented by Pauli matrix σ_y or **Y.** This gate maps $|0\rangle$ state to $i|1\rangle$ state and $|1\rangle$ state to $-\ |0\rangle$ state.
The matrix representation of Y Gate and its operation on$|0\rangle$and $|1\rangle$ are as follows

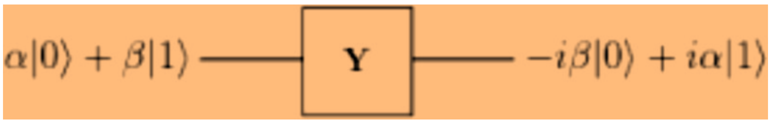
$$Y\ |0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle$$

$$Y\ |1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|0\rangle$$

Thus the Y-gate defines the transformation

$$(\alpha|0\rangle+\beta|1\rangle)=\alpha Y|0\rangle+\beta Y|1\rangle=-i\beta|0\rangle+i\alpha|1\rangle$$

The quantum Y gate is represented by



Truth table of Y gate

Input	Output
$\|0\rangle$	$i\|1\rangle$
$\|1\rangle$	$-i\|0\rangle$
$\alpha\|0\rangle+\beta\|1\rangle$	$-i\beta\|0\rangle+i\alpha\|1\rangle$

Pauli Z gate

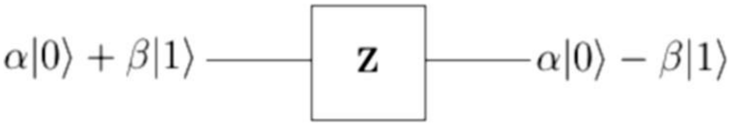
The Z-gate is represented by Pauli matrix σ_z or Z. Z gate leaves a $|0\rangle$ state unchanged but flips the sign of the $|1\rangle$ state to $-|1\rangle$.

The matrix representation and the operation of Z-gate on$|0\rangle$and$|1\rangle$ are as follows,

$$Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$Z\,|1\rangle = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} = -|1\rangle$$

The Pauli Z gate represented by,



Truth table

Input	Output
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$-\|1\rangle$
$\alpha\|0\rangle+\beta\|1\rangle$	$\alpha\|0\rangle-\beta\|1\rangle$

Hadamard gate:

The Hadamard gate (H-gate) is a truly quantum gate and is one of the most important gate in quantum computing. The H-gate is a single-qubit operation that maps the basis state$|0\rangle$ to $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$, $|1\rangle$ to $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$ and $\alpha\,|0\rangle + \beta|1\rangle$ to $\frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$.

The matrix representation is

$$H=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

the Hadanard gate satisfies unitary condition. $H^{\dagger}H=1$

The Hadamard gate can be represent by,

$$\alpha|0\rangle + \beta|1\rangle - H - \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$$

Truth table:

Input	Action of Hardmard gate	Output
$\|0\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$
$\|1\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$
$\alpha\,\|0\rangle + \beta\|1\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha+\beta\\ \alpha-\beta\end{bmatrix}$	$\frac{\alpha+\beta}{\sqrt{2}}\|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}\|1\rangle$

Phase gate or S gate

Phase gate or S gate is a quantum gate that turns a state $|0\rangle$ into $|0\rangle$ and state $|1\rangle$ in to $i|1\rangle$. It is also known as Z90 gate because it represents a 90-degree rotation around the z-axis in Bloch sphere. The S gate satisfies the unitary condition $S^{\dagger}. S = I$

Matrix representation $S=\begin{pmatrix}1 & 0\\0 & i\end{pmatrix}$

S gate Operation 0 and 1 states

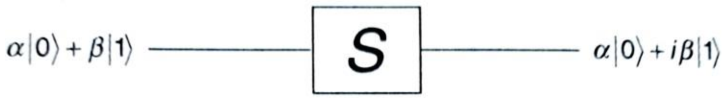
$$S|0\rangle = \begin{pmatrix}1 & 0\\0 & i\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = |0\rangle$$

$$S|1\rangle = \begin{pmatrix}1 & 0\\0 & i\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = i|1\rangle$$

The transformation of S gate is given by

$S|\psi\rangle = S(\alpha|0\rangle+\beta|1\rangle) = \alpha S|0\rangle+\beta S|1\rangle = \alpha|0\rangle+i\beta|1\rangle$

The Phase or S gate represented by



Truth table of S gate

Input	Output
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$i\|1\rangle$
$\alpha\|0\rangle+\beta\|1\rangle$	$\alpha\|0\rangle+i\beta\|1\rangle$

T gate or $\frac{\Pi}{8}$ gate

The T gate is represented by the matrix as follows.

$$T=\begin{bmatrix}1 & 0\\0 & e^{\frac{i\pi}{4}}\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & \frac{1+i}{\sqrt{2}}\end{bmatrix}$$

The T gate is related to S gate by the relation $S=T^2$

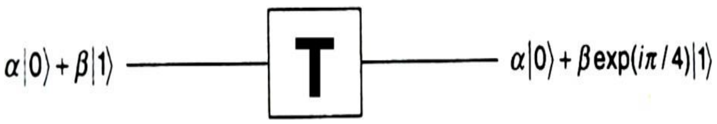
The operation of T gate on $|0\rangle$ and $|1\rangle$ is given by

$$T|0\rangle = \begin{bmatrix}1 & 0\\0 & \frac{1+i}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

Therefore $T|0\rangle = |0\rangle$

$$T|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1+i}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore $T|1\rangle = \frac{1+i}{\sqrt{2}}\ |1\rangle$



Truth table of T gate

Input	Output
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$e^{\frac{i\pi}{4}}\|1\rangle$
$\alpha\|0\rangle+\beta\|1\rangle$	$\alpha\|0\rangle+\beta e^{\frac{i\pi}{4}}\ \|1\rangle$

Multiple qubit gates

Multiple qubit gates operate on two or more input qubits. Usually one of them is a control qubit.

Controlled gates

A gate with operation of kind "If $\boldsymbol{A}$ is true then do $\boldsymbol{B}$" is called controlled gate. The qubit $|A\rangle$ is called Control qubit and $|B\rangle$ is the Target qubit. The target qubit is altered only when the control qubit is $|1\rangle$. The control qubit remains unaltered during the transformations.

Controlled Not gate or CNOT gate

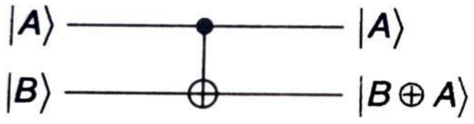
The **CNOT gate** or **controlled-bit-flip gate** is a quantum logic gate that is an essential component in the construction of a gate-based quantum computer. It can be used to entangle and disentangle bell states.

The matrix representation of CNOT gate is as follows

$$U_{CN} = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \\ 0\,0\,1\,0 \end{pmatrix}$$

The transformation could be expressed as,

$$|A,B\rangle \rightarrow |A,B \oplus A\rangle$$



CNOT gate

The operations of CNOT gate on the four inputs$|00\rangle$, $|01\rangle$, $|10\rangle and |11>$

Operation of CNOT gate for input $|00\rangle$

In this operation the input to the CNOT Gate is $|00\rangle$ and the control qubit is$|0\rangle$.Hence no change in the state of target qubit $|1\rangle$.

$$|00\rangle \rightarrow |00\rangle$$

Operation of CNOT gate for input $|01\rangle$

In this operation the input to the CNOT Gate is$|01\rangle$ and the control qubit is $|0\rangle$. Hence no change in the state of target qubit $|1\rangle$.

$$|01\rangle \rightarrow |01\rangle$$

Operation of CNOT gate for input $|10\rangle$

In this operation the input to the CNOT Gate is$|10\rangle$and the control qubit is $|1\rangle$. Hence the state of target qubit flips from $|0\rangle$ to $|1\rangle$.

$$|10\rangle \rightarrow |11\rangle$$

Operation of CNOT gate for input $|11\rangle$

In this operation the input to the CNOT Gate is$|11\rangle$and the control qubit is $|1\rangle$. Hence the state of target qubit flips from $|1\rangle$ to $|0\rangle$.

$$|11\rangle \rightarrow |10\rangle$$

Truth table of CNOT gate

Input	output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

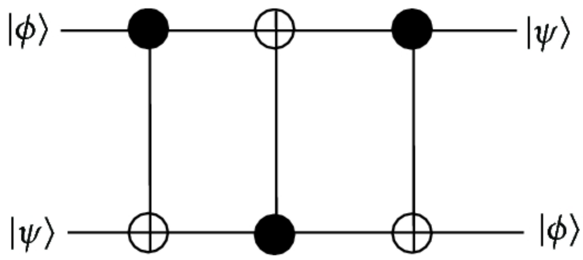
Swap gate

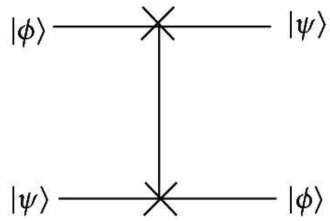
The SWAP gate is a two-qubit operation, expressed in basis states. The SWAP gate swaps the state of the two qubits involved in the operation.

The matrix representation of Swap gate is as follows,

$$U_{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The SWAP gate interchanges the input states say, $|\emptyset\rangle$ and $|\psi\rangle$. The schematic swap gate circuit symbol is as follows, which is equivalent to the combined circuit of 3 CNOT gates and the overall effect is that two input qubits are swapped at the output.





Gate	Input to gate	Output of gate
1	$\|a,b\rangle$	$\|a,a\oplus b\rangle$
2	$\|a,a\oplus b\rangle$	$\|b,a\oplus b\rangle$
3	$\|b,a\oplus b\rangle$	$\|b,a\rangle$

Truth table of SWAP gate

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|10\rangle$
$\|10\rangle$	$\|01\rangle$
$\|11\rangle$	$\|11\rangle$

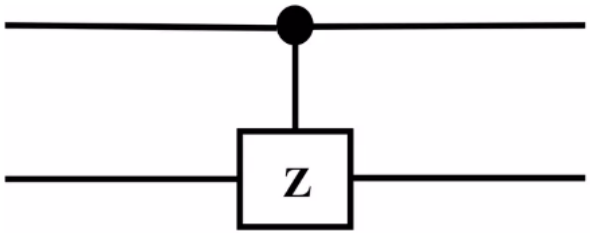
Controlled Z (CZ) gate

In Controlled Z gate, the operation of Z gate is controlled by a control qubit. If the control qubit is $|A\rangle = |1\rangle$ then only the Z gate transforms the target qubit $|B\rangle$ as per the Pauli-Z operation. The CZ gate is a two qubit operation and is specified by a matrix as follows,

The symbolic representation of CZ gate is as follows

$$U_Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Truth table of Controlled Z gate



Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|10\rangle$
$\|11\rangle$	$-\|11\rangle$

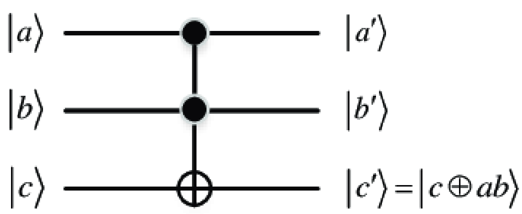
Toffoli gate (CCNOT) gate

The Toffoli gate also known as CCNOT gate. It is also known as the controlled-controlled-not gate that describes the action. It has inputs and outputs; the first two bits are control bits which are unaffected by the action of the Toffoli gate. The third is the target bit which is inverted if both the control bits are $|1\rangle$ else it remains unchanged. The Toffoli matrix is unitary. The Toffoli gate is its own inverse. It could be used for NAND gate simulation.

The matrix representation of Toffoli gate is as follows,

$$U_T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The circuit representation of Toffoli gate is as follows



Truth table of Toffoli gate

Input			Output		
$\|a\rangle$	$\|b\rangle$	$\|c\rangle$	$\|a'\rangle$	$\|b'\rangle$	$\|c'\rangle$
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	1	0	1	1	1
1	1	1	1	1	0