NAGARJUNA COLLEGE OF ENGINEERING AND TECHNOLOGY

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NAGARJUNA
COLLEGE OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

CALCULUS AND LINEAR ALGEBRA

(COURSE CODE 23MATS11)

MODULE-5

LINEAR ALGEBRA

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Module-5

LINEAR ALGEBRA

Elementary row transformation of a matrix, Rank of a matrix. Consistency and Solution of system of linear equations; Gauss-elimination method, Gauss-Jordan method and Approximate solution by Gauss-Seidel method. Eigen values and Eigenvectors-Rayleigh's power method to find the dominant Eigen value and Eigenvector.

RANK OF A MATRIX:

Minor of a matrix:

A minor of the matrix A is the determinant of order r obtained by selecting r rows and r columns of A, deleting all other rows and columns.

Definition of Rank:

A matrix A is said to be of rank r if (i) it has at least one non-zero minor of order r, and

(ii) Every minor of order higher than r vanishes

i.e., the rank of a matrix A is the largest order of any non-vanishing minor of the matrix A

The rank of a matrix A is denoted by $\rho(A)$.

Elementary Transformation of a matrix:

The following operations are known as elementary transformations.

(i)The interchange of any two rows or columns.

(ii)The multiplication of any row or column by a non-zero number.

(iii)The addition of a constant multiple of the elements of any row or column to the corresponding elements of any other row or column.

Echelon form:

A non-zero matrix A is said to be in row echelon form if it satisfies the following conditions.

(i) The leading entry of each row should be unity or a non-zero number.

(ii) All the entries below this leading entry should be zero.

(iii) The number of zeros appearing before the leading entry in each row is greater

than that appear in its previous row.

(iv) The zero rows must be appearing below non-zero row

Rank of the matrix by elementary transformations:

The rank of a matrix A is equal to the number of non-zero rows in echelon form of A.

Problems:

1. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ by reducing it to echelon form.

Solution: Given A = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

Using elementary row operations, $R_2 \rightarrow R_2 - R_1, \quad R_3 \rightarrow R_3 - 2R_1$.

$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$ $R_2 \rightarrow 1 - 1 = 0,\ 4 - 2 = 2,\ 2 - 3 = -1.$

$R_3 \rightarrow 2 - 2(1) = 0,\ 6 - 2(2) = 2,\ 5 - 2(3) = -1.$

Using elementary row operation, $R_3 \rightarrow R_3 - R_2$.

$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \rightarrow 2 - 2 = 0,\ -1 - (-1) = 0.$

This is the echelon form of the matrix A. In which the number of non-zero rows are two.
∴ Rank of A = 2. i.e., $\rho(A) = 2$.

2. Find the rank of the matrix $\begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$ by reducing it to echelon form.

Solution:

Given A = $\begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$ Using elementary row operation $R_1 \leftrightarrow R_2$, we get,

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 0 & 2 & 1 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

Using elementary row operations, $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1,\ R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

$R_2 \to 4 - 2(2) = 0,\ 0 - 2(1) = -2,\ 2 - 2(3) = -4,\ 1 - 2(4) = -7$

$R_3 \to 2 - 2 = 0,\ 3 - 1 = 2,\ 4 - 3 = 1,\ 7 - 4 = 3$

$R_4 \to 2 - 2 = 0,\ 3 - 1 = 2,\ 1 - 3 = -2,\ 4 - 4 = 0.$

Using operations, $R_3 \to R_3 + R_2,\ R_4 \to R_4 + R_2.$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & -6 & -7 \end{bmatrix}$$

$R_3 \to 2 + (-2) = 0,\ 1 + (-4) = -3,\ 3 + (-7) = -4.$

$R_4 \to 2 + (-2) = 0,\ -2 + (-4) = -6,\ 0 + (-7) = -7.$

Using operations, $R_4 \to R_4 - 2R_3$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R_4 \to -6 - 2(-3) = 0,\ -7 - 2(-4) = 1.$$

This is the echelon form of the matrix A. In which the number of non-zero rows are four. ∴ Rank of A = 4. i.e., ρ(A) = 4.

3. Find the rank of the matrix $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **by reducing it to echelon form.**

Solution: **Given** $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Using elementary row operation $R_1 \leftrightarrow R_2$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Using elementary row operations, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - R_1$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

Using elementary row operations, $R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_2$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the echelon form of the matrix A. In which the number of non-zero rows are two. ∴ Rank of A = 2. i.e., ρ(A) = 2.

4. Find the rank of the matrix $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$ by reducing it to echelon form.

Solution: Given $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

Using elementary row operations, $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$, we get,

$$A \sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Using elementary row operation $R_2 \to \frac{-1}{6} R_2$, we get,

$$A \sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Using elementary row operation $R_3 \to R_3 + 2R_2$, we get,

$$A \sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the echelon form of the matrix A. In which the number of non-zero rows are two. ∴ Rank of A = 2. i.e., ρ(A) = 2.

5. Find the rank of the matrix $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ by reducing it to echelon form.

Solution:

Given $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Using elementary row operations, $R_1 \leftrightarrow R_2$, **we get,**

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Using elementary row operation $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1,\ R_4 \to R_4 - 6R_1,$ **we get**

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Using elementary row operation $R_3 \to 5R_3 - 4R_2,\ R_4 \to 5R_4 - 9R_2,$ **we get**

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{bmatrix}$$

Using elementary row operation $R_4 \to R_4 - R_3,$ **we get**

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the echelon form of the matrix A. In which the number of non-zero rows are three. ∴ Rank of A = 3. i.e., ρ(A) = 3.

HOME WORK:

1. Find the rank of the matrix $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$ by reducing it to echelon form.

2. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & 8 \end{bmatrix}$ by reducing into echelon form.

3. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 0 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ by reducing into echelon form.

4. Find the rank of the matrix $\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$ by reducing into echelon form.

5. Find the rank of the matrix $\begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$ by reducing into echelon form.

Consistency and solution of linear equations:

Consider the system of m linear equations

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$ $\quad\rightarrow (1)$

……………………………………….

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$

Where $a_{ij}'s$ *and* $b_i's$ **are constants**

If $b_1, b_2, \ldots.. b_m$ **are all zero, the system is said to be homogenous. The set of values** $x_1, x_2, \ldots x_n$ **which satisfy all the equations simultaneously is called solution of the system of equations. A system of linear equations is said to be consistent if it possesses a solution otherwise the system is said to be inconsistent**

The above system of equations can be written in the matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$x_1 = x_2 = x_3 = \cdots x_n = 0$ is obviously a solution of the homogenous system of equations and is called trivial solution. If at least one $x_i(i = 1, 2, .. n)$ is not equal to zero then it is called non-trivial solution.

The concept of the rank of a matrix helps us to conclude

(i) Whether the system is consistent or not

(ii) Whether the system possess unique solution or many solutions

Conditions for consistency and types of solutions

The system of equations represented by the matrix form AX=B is consistent if $\rho(A) = \rho[A{:}B]$ where $[A{:}B]$ is augmented matrix

Suppose $\rho(A) = \rho[A{:}B] = r$, then the condition for various types of solution are as follows:

(i) Unique solution: $\rho(A) = \rho[A{:}B] = r = n$, where n is the number of unknowns.

(ii) Infinite solution: $\rho(A) = \rho[A{:}B] = r < n$. In this case (n-r) unknowns can take arbitrary values.

(iii) If $\rho(A) \neq \rho[A{:}B]$ implies that the system is inconsistent.

Problems:

1. Test the consistency and Solve

$$x + y + z = 6 \,, x - y + 2z = 5 \,, 3x + y + z = 8$$

Solution: Let $[A{:}B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 2 & : & 5 \\ 3 & 1 & 1 & : & 8 \end{bmatrix}$ be the augmented matrix

Perform the row operation to reduce the above matrix to echelon form

$$R_2 \to R_2 - R_1 \,, R_3 \to R_3 - 3R_1$$

$$[A{:}B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & -2 & -2 & : & -10 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & 0 & -3 & : & -9 \end{bmatrix}$$

$\rho(A) = 3,\ \rho[A:B] = 3\ , n = 3$

i.e., $\rho(A) = \rho[A:B] = 3, (r = n = 3)$

∴ **The given system of equation is consistent and will have Unique solution.**

Let us convert the prevailing form of [A : B] into set of equations

$x + y + z = 6$ …….. (1)

$-2y + z = -1$…….. (2)

$-3z = -9 \Rightarrow \boxed{z = 3}$

(2)=> $-2y + 3 = -1 \Rightarrow -2y = -4 \Rightarrow \boxed{y = 2}$

(1)=> $x + 2 + 3 = 6 \Rightarrow \boxed{x = 1}$

∴ $x = 1\ , y = 2\ , z = 3$ **is a unique solution.**

2. Test the consistency and Solve

$$x + 2y + 3z = 14\ ,\ 4x + 5y + 7z = 35\ , 3x + 3y + 4z = 21$$

Solution: Let $[A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 4 & 5 & 7 & : & 35 \\ 3 & 3 & 4 & : & 21 \end{bmatrix}$ **be the augmented matrix**

Perform the row operation to reduce the above matrix to echelon form

$R_2 \to R_2 - 4R_1\ , R_3 \to R_3 - 3R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -3 & -5 & : & -21 \\ 0 & -3 & -5 & : & -21 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -3 & -5 & : & -21 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\rho(A) = 2,\ \rho[A:B] = 2\ , \rho(A) = \rho[A:B] = 2 < 3\ (n = 3)$ **the system is consistent and will have infinite solution. Here** $n - r = 3 - 2 = 1$ **and hence one of the variable can take arbitrary value**

i.e., $z = k$ **be arbitrary**

$x + 2y + 3k = 14$ …. (1)

$-3y-5k=-21 \Rightarrow 3y=21-5k \Rightarrow y=7-\frac{5k}{3}$

(1)=> $x+2\left(7-\frac{5k}{3}\right)+3k=14 \Rightarrow 3x-k=0 \Rightarrow x=\frac{k}{3}$

$\therefore x=\frac{k}{3}, y=7-\frac{5k}{3}, z=k$ represents infinite solution.

3. Test the consistency and Solve

$$x-4y+7z=14,\ 3x+8y-2z=13,\ 7x-8y+26z=5$$

Solution: Let $[A:B]=\begin{bmatrix}1 & -4 & 7 & : & 14\\ 3 & 8 & -2 & : & 13\\ 7 & -8 & 26 & : & 5\end{bmatrix}$ be the augmented matrix

Perform the row operation to reduce the above matrix to echelon form

$R_2 \to R_2-3R_1, R_3 \to R_3-7R_1$

$$[A:B]\sim\begin{bmatrix}1 & -4 & 7 & : & 14\\ 0 & 20 & -23 & : & -29\\ 0 & 20 & -23 & : & -93\end{bmatrix}$$

$R_3 \to R_3-R_2$

$$[A:B]\sim\begin{bmatrix}1 & -4 & 7 & : & 14\\ 0 & 20 & -23 & : & -29\\ 0 & 0 & 0 & : & -64\end{bmatrix}$$

$\rho(A)=2, \rho[A:B]=3, \rho(A)\neq\rho[A:B]$ the system is inconsistent.

4. Investigate the values of λ and μ so that the equations

$2x+3y+5z=9, 7x+3y-2z=8, 2x+3y+\lambda z=\mu$ may have

(i) Unique solution (iii) Infinite solution (iii) No solution

Solution: Let $[A:B]=\begin{bmatrix}2 & 3 & 5 & : & 9\\ 7 & 3 & -2 & : & 8\\ 2 & 3 & \lambda & : & \mu\end{bmatrix}$ be the augmented matrix

Perform the row operation to reduce the above matrix to echelon form

$R_2 \to 2R_2-7R_1,\ R_3 \to R_3-R_1$

$$[A:B]\sim\begin{bmatrix}2 & 3 & 5 & : & 9\\ 0 & -15 & -39 & : & -47\\ 0 & 0 & \lambda-5 & : & \mu-9\end{bmatrix}$$

a) Unique solution: we must have $\rho(A)=\rho[A:B]=3$, $\rho(A)$ will be 3 if $\lambda-5\neq 0$ or $\lambda\neq 5$ irrespective of values of μ $\rho[A:B]$ will also be 3
$\therefore$ the system will have Unique solution if $\lambda\neq 5$

b) Infinite solution: here we have n = 3, we need $\rho(A) = \rho[A: B] = r < 3$ we must have $r = 2$
$\rho(A) = \rho[A: B] = 2$ only when last row of [A : B] is completely zero and it is possible only if $\lambda - 5 = 0$ and $\mu - 9 = 0$

∴ the system will have Infinite solution if $\lambda = 5$ and $\mu = 9$

c) No solution: we must have $\rho(A) \neq \rho[A: B]$. If $\lambda = 5$ we obtain $\rho(A) = 2$.If we impose $\mu - 9 \neq 0$ then $\rho[A: B] = 3$

∴ the system will have No solution if $\lambda = 5$ and $\mu \neq 9$

5. Find for what values of k the system of equation

$x + y + z = 1$, $x + 2y + 4z = k$, $x + 4y + 10z = k^2$ possesses a solution. Solve completely in each case.

Solution: Let $[A: B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 4 & : & k \\ 1 & 4 & 10 & : & k^2 \end{bmatrix}$ be the augmented matrix.

Perform the row operation to reduce the above matrix to echelon form

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$[A: B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & k-1 \\ 0 & 3 & 9 & : & k^2 - 1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & k-1 \\ 0 & 0 & 0 & : & k^2 - 3k + 2 \end{bmatrix}$$

Since $\rho(A) = 2$ and the system to be consistent we must have $\rho[A: B] = 2$ this is possible only if $k^2 - 3k + 2 = 0 \Rightarrow k = 1, 2$. Hence the system possesses a solution if $k = 1, 2$

Since $\rho(A) = \rho[A: B] = 2 < 3$. $n - r = 3 - 2 = 1$ for the cases $k = 1, 2$ the system will have infinite solution.

Case i) $k = 1$ the system of equations are

$z = k_1$ be arbitrary

$y + 3k_1 = 0 \Rightarrow y = -3k_1$

$x - 3k_1 + k_1 = 1 \Rightarrow x = 1 + 2k_1$

Case ii) $k = 2$ the system of equations are

$z = k_2$ be arbitrary

$y + 3k_2 = 1 \Rightarrow y = 1 - 3k_2$

$$x + (1 - 3k_2) + k_2 = 1 => x - 2k_2 = 0 => x = 2k_2$$

Thus $x = 1 + 2k_1 , y = -3k_1 , z = k_1$ and $x = 1 + 2k_1 , y = -3k_1 , z = k_1$ give all the solution of the given system of equations.

HOME WORK

1. Test for consistency and Solve

$$5x_1 + x_2 + 3x_3 = 20 , 2x_1 + 5x_2 + 2x_3 = 18 , 3x_1 + 3x_2 + x_3 = 14$$

2. Test for consistency and Solve

$$x + 2y + 2z = 1 , 2x + y + z = 2 , 3x + 2y + 2z = 3 , y + z = 0$$

3. Show that the following system of equations does not possess any solution

$$5x + 3y + 7z = 5 , 3x + 26y + 2z = 9 , 7x + 20y + 10z = 5$$

4. Investigate the values of λ and μ such that the system of equations

$x + y + z = 6 , x + 2y + 3z = 10 , x + 2y + \lambda z = \mu$ **may have**

a) **Unique solution b) Infinite solution c) No solution**

Solution of system of linear equations:

Gauss Elimination Method:

Consider the system of equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{array}\right\} \quad \text{..........(1)}$$

The system (1) is equivalent to the matrix equation AX = B ....(2)

$$\text{Where} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Consider the augmented matrix [A : B].

$$\text{We have } [A : B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & : b_1 \\ a_{21} & a_{22} & a_{23} & : b_2 \\ a_{31} & a_{32} & a_{33} & : b_3 \end{bmatrix}$$

We reduce this matrix to the echelon form by using elementary row transformations.

Step 1: We use the element a_{11} $(\neq 0)$ to make the elements a_{21} and a_{31} as zero by elementary row transformations. Then [A : B] becomes,

$$[A : B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & : b_1 \\ 0 & a'_{22} & a'_{23} & : b'_2 \\ 0 & a'_{32} & a'_{33} & : b'_3 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Step 2: We use the element a_{22}' $(\neq 0)$ to make the elements a_{32}' as zero by elementary row transformations. Then [A : B] becomes,

$$[A : B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & : b_1 \\ 0 & a'_{22} & a'_{23} & : b'_2 \\ 0 & 0 & a''_{33} & : b''_3 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

From (4) the given system of equations is equivalent to the system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 = b_2'$$

$$a''_{33}x_3 = b''_3$$

From the last equation we get, $x_3 = \frac{b''_3}{a''_{33}}$ and by back substitution we get the values of x_2 and x_1. This method is known as Gauss elimination method.

Problems:

1. **Solve the system of linear equations $2x_1 + 4x_2 + x_3 = 3$, $3x_1 + 2x_2 - 2x_3 = -2$, $x_1 - x_2 + x_3 = 6$ using Gauss elimination method.**

Solution: The augmented matrix of the given system is

$$[A{:}\,B] = \begin{bmatrix} 2 & 4 & 1 & : & 3 \\ 3 & 2 & -2 & : & -2 \\ 1 & -1 & 1 & : & 6 \end{bmatrix}$$ Using elementary row operation, $R_1 \leftrightarrow R_3$, we get,

$$[A{:}\,B] \sim \begin{bmatrix} 1 & -1 & 1 & : & 6 \\ 3 & 2 & -2 & : & -2 \\ 2 & 4 & 1 & : & 3 \end{bmatrix}$$

Using elementary row operation, $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$[A:B] \sim \begin{bmatrix} 1 & -1 & 1 & : & 6 \\ 0 & 5 & -5 & : & -20 \\ 0 & 6 & -1 & : & -9 \end{bmatrix}$ R_2 → 3 – 3(1) = 0, 2 – 3(-1) = 5, - 2 – 3(1)= - 5,

- 2 – 3(6) = - 20. R_3 →2 – 2(1) = 0, 4 – 2(-1) = 6, 1 – 2(1) = - 1, 3 – 2(6) = - 9.

Using elementary row operation, $R_2 \to \frac{1}{5} R_2$, we get,

$$[A:B] \sim \begin{bmatrix} 1 & -1 & 1 & : & 6 \\ 0 & 1 & -1 & : & -4 \\ 0 & 6 & -1 & : & -9 \end{bmatrix}$$

Using elementary row operation, $R_3 \to R_3 - 6R_2$, we get,

$[A:B] \sim \begin{bmatrix} 1 & -1 & 1 & : & 6 \\ 0 & 1 & -1 & : & -4 \\ 0 & 0 & 5 & : & 15 \end{bmatrix}$ R_3 →6 – 6(1) = 0, - 1 – 6(-1) = 5, - 9 – 6(- 4) = 15.

Hence we get, $x_1 - x_2 + x_3 = 6$

$$x_2 - x_3 = -4$$

$$5x_3 = 15 \quad \therefore x_3 = 3$$

By back substitution, we get, $x_2 = -1$ **and** $x_1 = 2$

$\therefore$ **Solution is** $x_1 = 2$, $x_2 = -1$, $x_3 = 3$.

2. Find the Solution of the system of linear equations using Gauss elimination method:

$2x - y + 3z = 9, \quad x + y + z = 6, \quad x - y + z = 2.$

Solution: **The augment matrix of the system is**

$$[A:B] = \begin{bmatrix} 2 & -1 & 3 & : & 9 \\ 1 & 1 & 1 & : & 6 \\ 1 & -1 & 1 & : & 2 \end{bmatrix}$$

Using elementary row operation, $R_1 \leftrightarrow R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 2 & -1 & 3 & : & 9 \\ 1 & -1 & 1 & : & 2 \end{bmatrix}$$

Using elementary row operation, $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -3 & 1 & : & -3 \\ 0 & -2 & 0 & : & -4 \end{bmatrix}$$

Using elementary row operation, $R_3 \rightarrow 3R_3 - 2R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -3 & 1 & : & -3 \\ 0 & 0 & -2 & : & -6 \end{bmatrix}$$

Hence we get, $x + y + z = 6$

$$-3y + z = -3$$

$$-2z = -6. \quad \therefore \quad z = 3$$

By back substitution, we get, $y = 2$ and $x = 1$

$\therefore$ Solution is $x = 1, \ y = 2, \ z = 3.$

3. Find the Solution of the system of linear equations using Gauss elimination method:

$$x + 4y - z = -5, \quad x + y - 6z = -12, \quad 3x - y - z = 4.$$

Solution: The augmented matrix of the given system is

$$[A:B] = \begin{bmatrix} 1 & 4 & -1 & : & -5 \\ 1 & 1 & -6 & : & -12 \\ 3 & -1 & -1 & : & 4 \end{bmatrix}$$

Using elementary row operation, $R_2 \rightarrow R_2 - R_1, \quad R_3 \rightarrow R_3 - 3R_1$ we get,

$$[A:B] \sim \begin{bmatrix} 1 & 4 & -1 & : & -5 \\ 0 & -3 & -5 & : & -7 \\ 0 & -13 & 2 & : & 19 \end{bmatrix}$$ $R_2 \rightarrow$ 1 – 1 = 0, 1 – 4 = - 3, - 6 + 1 = - 5, - 12 + 5 = - 7.

$R_3 \rightarrow$ 3 – 3(1) = 0, - 1 – 3(4) = - 13, - 1 – 3(-1) = 2, 4 – 3(-5) = 19.

Using elementary row operation, $R_3 \rightarrow 3R_3 - 13R_2$ we get,

$$[A:B] \sim \begin{bmatrix} 1 & 4 & -1 & : & -5 \\ 0 & -3 & -5 & : & -7 \\ 0 & 0 & 71 & : & 148 \end{bmatrix}$$

$R_3 \rightarrow$ 3(-13) – 13(-3) = 0, 3(2) – 13(-5) =71, 3(19) – 13(-7) = 148.

Hence we get, $x + 4y - z = -5$

$$-3y - 5z = -7$$

$$71z = 148 \quad \therefore z = 2.0845$$

By back substitution, we get, $y = -1.1408$ and $x = 1.6479$

∴ **Solution is $x = 1.6479, \quad y = -1.1408, \quad z = 2.0845.$**

HOME WORK:

1. Solve the following system of equations $x + y + z = 9, \; x - 2y + 3z = 8$ and $2x + y - z = 3$, by Using Gauss elimination method.
2. Solve $2x + y + 4z = 12$, $4x + 11y - z = 33$, $8x - 3y + 2z = 20$ by using Gauss elimination method.
3. **Find the Solution of system of linear equations** $2x + 2y + z = 12$, $3x + 2y + 2z = 8$, $5x + 10y - 8z = 10$ by using Gauss Elimination Method.
4. **Find the Solution of system of linear equations** $x + y + z = 9$, $2x - 3y + 4z = 13$, $3x + 4y + 5z = 40$, by using Gauss Elimination Method.
5. **Find the Solution of system of linear equations** $2x + y + z = 10$, $3x + 2y + 3z = 18$, $x + 4y + 9z = 16$, by using Gauss Elimination Method.
6. **Find the Solution of system of linear equations** $2x_1 + x_2 + x_3 = 10$, $3x_1 + 2x_2 + 3x_3 = 18$, $x_1 + 4x_2 + 9x_3 = 16$, by using Gauss Elimination Method.

Solution of a system of non-homogenous equations

Gauss-Jordan method:

This method aims in reducing the co-efficient matrix A to a diagonal matrix

Step 1: This step is same as in Gauss Elimination method.

Step 2: we use the elements $a'_{22} (\neq 0)$ to make the element a_{12} *and* a'_{32} by elementary row transformation.

$$[A:B] = \begin{bmatrix} a_{11} & 0 & a'_{13} & : & b'_1 \\ 0 & a'_{22} & a'_{23} & : & b'_2 \\ 0 & 0 & a''_{33} & : & b''_{33} \end{bmatrix}$$

Step 3: $a''_{33} (\neq 0)$ to make the elements a'_{22}, a'_{13} zero

$$[A:B] = \begin{bmatrix} a_{11} & 0 & 0 & : & b_1'' \\ 0 & a_{22}' & 0 & : & b_2'' \\ 0 & 0 & a_{33}'' & : & b_{33}'' \end{bmatrix}$$

The matrix A is reduced to the diagonal form

$\therefore\ a_{11}x_1 = b_1''\ , a_{22}'x_2 = b_2''\ , a_{33}''x_3 = b_3''$

$\therefore x_1, x_2, x_3$ **being the exact solution.**

Problems:

1. Apply Gauss-Jordan method to solve the equation

$2x_1 + x_2 + 3x_3 = 1\ , 4x_1 + 4x_2 + 7x_3 = 1\ \ , 2x_1 + 5x_2 + 9x_3 = 3$

Solution: $[A:B] = \begin{bmatrix} 2 & 1 & 3 & : & 1 \\ 4 & 4 & 7 & : & 1 \\ 2 & 5 & 9 & : & 3 \end{bmatrix}$

$R_2 \to R_2 - 2R_1\ ,\ \ R_3 \to R_3 - R_1$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 3 & : & 1 \\ 0 & 2 & 1 & : & -1 \\ 0 & 4 & 6 & : & 2 \end{bmatrix}$$

$R_1 \to 2R_1 - R_2\ ,\ \ R_3 \to R_3 - 2R_2$

$$[A:B] \sim \begin{bmatrix} 4 & 0 & 5 & : & 3 \\ 0 & 2 & 1 & : & -1 \\ 0 & 0 & 4 & : & 4 \end{bmatrix}$$

$R_3 \to \frac{R_3}{4}$

$$[A:B] \sim \begin{bmatrix} 4 & 0 & 5 & : & 3 \\ 0 & 2 & 1 & : & -1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$R_1 \to R_1 - 5R_3\ ,\ \ R_2 \to R_2 - 2R_3$

$$[A:B] \sim \begin{bmatrix} 4 & 0 & 0 & : & -2 \\ 0 & 2 & 0 & : & -2 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

Hence we $x_3 = 1\ , 2x_2 = -2 => x_2 = -1, 4x_1 = -2 => x_1 = -0.5$

$x_1 = -0.5, x_2 = -1\ , x_3 = 1$ is the required solution.

2. Apply Gauss-Jordan method to solve the system of equation

$2x + 5y + 7z = 52 , 2x + y - z = 0 , x + y + z = 9$

Solution: $[A: B] = \begin{bmatrix} 2 & 5 & 7 & : & 52 \\ 2 & 1 & -1 & : & 0 \\ 1 & 1 & 1 & : & 9 \end{bmatrix}$

$R_1 \leftrightarrow R_3$

$[A: B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 1 & -1 & : & 0 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$

$R_2 \to R_2 - 2R_1 , \; R_3 \to R_3 - 2R_1$

$[A: B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 3 & 5 & : & 34 \end{bmatrix}$

$R_1 \to R_2 + R_1 , \; R_3 \to 3R_2 + R_3$

$[A: B] \sim \begin{bmatrix} 1 & 0 & -2 & : & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & -4 & : & -20 \end{bmatrix}$

$R_3 \to \frac{R_3}{-4}$

$[A: B] \sim \begin{bmatrix} 1 & 0 & -2 & : & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$

$R_1 \to 2R_3 + R_1 , \; R_2 \to 3R_3 + R_2$

$[A: B] \sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & -1 & 0 & : & -3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$

Hence we have $z = 5 , -y = -3 => y = 3 , x = 1$

$\therefore x = 1, y = 3 , z = 5$ **is the required solution.**

HOME WORK

1. Solve $10x - 7y + 3z + 5u = 6 , -6x + 8y - z - 4u = 5 , 3x + y + 4z + 11u = 2 ,$

$5x - 9y - 2z + 4u = 7$ **by Gauss Jordan method**

2. Solve the following system of equation by Gauss Jordan method

$x_1 + x_2 + x_3 + x_4 = 2 , 2x_1 - x_2 + 2x_3 - x_4 = -5 \; , 3x_1 + 2x_2 + 3x_3 + 4x_4 = 7 ,$

$x_1 - 2x_2 - 3x_3 + 2x_4 = 5$

Gauss –Seidel Method:

Consider the system of equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\}$$ **………. (1), in which the coefficients of the leading**

diagonal are numerically larger than the other coefficients. Such a system is called the diagonally dominant system. If the system is not diagonally dominant, then rearrange the system as diagonally dominant. Then write the system (1) as

$$x_1 = \frac{1}{a_{11}}[b_1 - a_{12}x_2 - a_{13}x_3]$$

$$x_2 = \frac{1}{a_{22}}[b_2 - a_{21}x_1 - a_{23}x_3]$$

$$x_3 = \frac{1}{a_{33}}[b_3 - a_{31}x_1 - a_{32}x_2]$$

We start with the initial approximation $x_1 = x_1^{(0)}, \ x_2 = x_2^{(0)}, \ x_3 = x_3^{(0)}$,

then the first approximation is as follows.

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right]$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}\right]$$

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right]$$

Similarly, the second approximation is as follows.

$$x_1^{(2)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)}\right]$$

$$x_2^{(2)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(2)} - a_{23}x_3^{(1)}\right]$$

$$x_3^{(2)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(2)} - a_{32}x_2^{(2)}\right]$$

Continue this process until to get the solution to the desired degree of accuracy.

Problems:

1. **Find the Solution of system of linear equation using Gauss–Seidel method.**

$$20x + y - 2z = 17, \quad 3x + 20y - z = -18, \quad 2x - 3y + 20z = 25.$$

Solution:

The given system is diagonally dominant and hence we first write them in the following form

$$x = \frac{1}{20}[17 - y + 2z]$$

$$y = \frac{1}{20}[-18 - 3x + z]$$

$$z = \frac{1}{20}[25 - 2x + 3y]$$

We start with the initial solution $x = 0, \; y = 0, \; z = 0.$

First iteration:

$$x^{(1)} = \frac{17}{20} = 0.85$$

$$y^{(1)} = \frac{1}{20}[-18 - 3(0.85)] = -1.0275$$

$$z^{(1)} = \frac{1}{20}[25 - 2(0.85) + 3(-1.0275)] = 1.0109$$

Second iteration:

$$x^{(2)} = \frac{1}{20}[17 - (-1.0275) + 2(1.0109)] = 1.0025$$

$$y^{(2)} = \frac{1}{20}[-18 - 3(1.0025) + 1.0109] = -0.9998$$

$$z^{(2)} = \frac{1}{20}[25 - 2(1.0025) + 3(-0.9998)] = 0.9998$$

Third iteration:

$$x^{(3)} = \frac{1}{20}[17 - (-0.9998) + 2(0.9998)] = 1.0000$$

$$y^{(3)} = \frac{1}{20}[-18 - 3(1) + 0.9998] = -1.0000$$

$$z^{(3)} = \frac{1}{20}[25 - 2(1) + 3(-1)] = 1.0000$$

Fourth iteration:

$x^{(4)} = \frac{1}{20}[17 - (-1) + 2(1)] = 1.0000$

$y^{(4)} = \frac{1}{20}[-18 - 3(1) + 1] = -1.0000$

$z^{(4)} = \frac{1}{20}[25 - 2(1) + 3(-1)] = 1.0000$

Thus the solution is $x = 1$, $y = -1$, $z = 1$.

2. Solve the equations $9x - y + 2z = 9$, $x + 10y - 2z = 15$, $-2x + 2y + 13z = 17$ using Gauss–Seidel method by taking (1, 1, 1) as initial approximate solution

Solution: **The equations are diagonally dominant and hence we first write them in the following form**

$x = \frac{1}{9}[9 + y - 2z]$

$y = \frac{1}{10}[15 - x + 2z]$

$z = \frac{1}{13}[17 + 2x - 2y]$

We start with the initial approximate solution $x = 1$, $y = 1$, $z = 1$.

First iteration:

$x^{(1)} = \frac{1}{9}(8) = 0.8889$

$y^{(1)} = \frac{1}{10}[15 - (0.8889) + 2(1)] = 1.6111$

$z^{(1)} = \frac{1}{13}[17 + 2(0.8889) - 2(1.6111)] = 1.1966$

Second iteration:

$x^{(2)} = \frac{1}{9}[9 + (1.6111) - 2(1.1966)] = 0.9131$

$y^{(2)} = \frac{1}{10}[15 - 0.9131 + 2(1.1966)] = 1.6480$

$z^{(2)} = \frac{1}{13}[17 + 2(0.9131) - 2(1.6480)] = 1.1946$

Third iteration:

$x^{(3)} = \frac{1}{9}[9 + (1.6480) - 2.3892] = 0.9176$

$y^{(3)} = \frac{1}{10}[15 + 2.3892 - 0.9176] = 1.6472$

$z^{(3)} = \frac{1}{13}[17 + 1.8352 - 3.2944] = 1.1954$

Fourth iteration:

$x^{(4)} = \frac{1}{9}[9 + (1.6472) - 2.3908] = 0.9174$

$y^{(4)} = \frac{1}{10}[15 + 2.3908 - 0.9174] = 1.6473$

$z^{(4)} = \frac{1}{13}[17 + 1.8348 - 3.2946] = 1.1954$

∴ solution is x=0.9174, y = 1.6473, z = 1.1954.

3. Solve the equations $\begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -6 \end{bmatrix}$ **using Gauss–Seidel method.**

Solution: **The system of equation is**

$5x - y + 0z = 9, \quad -x + 5y - z = 4, \quad 0x - y + 5z = -6.$

The equations are diagonally dominant and hence we first write them in the following form

write them in the following form

$x = \frac{1}{5}[9 + y]$

$y = \frac{1}{5}[4 + x + z]$

$z = \frac{1}{5}[-6 + y]$

We start with the trial solution $x = 0, \quad y = 0, \quad z = 0.$

First iteration:

$x^{(1)} = \frac{1}{5}[9 + 0] = 1.8$

$y^{(1)} = \frac{1}{5}[4 + 1.8 + 0] = 1.16$

$z^{(1)} = \frac{1}{5}[-6 + 1.16] = -0.968$

Second iteration:

$\mathbf{x^{(2)} = \frac{1}{5}[9 + 1.16] = 2.032}$

$\mathbf{y^{(2)} = \frac{1}{5}[4 + 2.032 - 0.968] = 1.0128}$

$\mathbf{z^{(2)} = \frac{1}{5}[-6 + 1.0128] = -0.9974}$

Third iteration:

$\mathbf{x^{(3)} = \frac{1}{5}[9 + 1.0128] = 2.0026}$

$\mathbf{y^{(3)} = \frac{1}{5}[4 + 2.00256 \ - 099744] = 1.0010}$

$\mathbf{z^{(3)} = \frac{1}{5}[-6 + 1.001024] = -0.9998}$

Fourth iteration:

$\mathbf{x^{(4)} = 2.0002,\ \ y^{(4)} = 1.0001,\ \ z^{(4)} = -1}$

Fifth iteration:

$\mathbf{x^{(5)} = 2.0000,\ \ y^{(5)} = 1.0000,\ \ z^{(4)} = -1}$

Thus $\mathbf{x = 2,\ y = 1,\ z = -1}$ is the required solution.

HOME WORK:

1. **Find the Solution of system of linear equation $\mathbf{2x + y + 6z = 9}$,**
   $\mathbf{8x + 3y + 2z = 13,\ \ x + 5y + z = 7}$ using Gauss seidel Method.
2. **Find the Solution of system of linear equation $\mathbf{28x_1 + 4x_2 - x_3 = 32}$,**
   $\mathbf{x_1 + 3x_2 + 10x_3 = 24,\ \ 2x_1 + 17x_2 + 4x_3 = 35}$ using Gauss seidel Method.
3. **Find the Solution of system of linear equation $\mathbf{10x + y + z = 12}$,**
   $\mathbf{2x + 10y + z = 13,\ \ 2x + 2y + 10z = 14}$ using Gauss seidel Method.
   Take (0, 0, 0) as initial approximate solution.
4. **Use the Gauss-Seidel method to solve the system 27x + 6y – z = 85,**
   6x + 15y +2z = 72 , x + y + 54z = 110. Take x = 2, y = 3, z = 2 as an initial

approximation ,carry out three iterations.

5. **Solve $2x + y + 6z = 9$, $8x + 3y + 2z = 13$, $x + 5y + z = 7$ by using Gauss Seidel Method. Take (1, 0, 0) as initial approximate solution.**

Eigen values and Eigen vectors of a square matrix:

Given A is a square matrix of order n. If there exists a scalar λ and non-zero column matrix X such that AX = λX, then λ is called an eigen value of A and X is called an eigen vector corresponding to the eigen value λ of A.

Working procedure to obtain Eigen values and Eigen vectors:

1. **If A is a square matrix of order n and λ is a scalar then the equation $|A - \lambda I| = 0$ is called the characteristic equation of A. The roots of this equation are called eigen values or characteristic roots or latent roots of A.**
2. **Solving the equation $|A - \lambda I| = 0$, we obtain the eigen values of A.**
3. **Consider a system of equations $[A - \lambda I]X = 0$. On solving this system of equations, we obtain the eigen vector X corresponding to each of the eigen value λ.**

Problems:

1. Find the eigen value and the eigen vectors of the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Solution: **Let** $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, **then the characteristic equation of A is** $|A - \lambda I| = 0$.

$$\therefore \begin{vmatrix} (8-\lambda) & -6 & 2 \\ -6 & (7-\lambda) & -4 \\ 2 & -4 & (3-\lambda) \end{vmatrix} = 0$$

$$\therefore (8-\lambda)[(7-\lambda)(3-\lambda) - 16] - (-6)[-6(3-\lambda) + 8] + 2[24 - 2(7-\lambda)] = 0$$

$$\therefore (8-\lambda)[5 - 10\lambda + \lambda^2] + 6[6\lambda - 10] + 2[10 + 2\lambda] = 0$$

$$\therefore 40 - 80\lambda + 8\lambda^2 - 5\lambda + 10\lambda^2 - \lambda^3 + 36\lambda - 60 + 20 + 4\lambda = 0$$

$\therefore -\lambda^3 + 18\lambda^2 - 45\,\lambda = 0 \quad \therefore\; \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\therefore \lambda(\lambda^2 - 18\,\lambda + 45) = 0 \quad \therefore \lambda(\lambda - 3)(\;\lambda - 15) = 0$

$\therefore\; \lambda = 0, \quad 3, \quad 15$ **are the eigen values of A.**

Consider the equation $[A - \lambda I]X = 0$. Where X = (x, y, z)^T is the eigen vector corresponding to λ.

$$\left.\begin{aligned} (8-\lambda)x - 6y + 2z &= 0 \\ -6x + (7-\lambda)y - 4z &= 0 \\ 2x - 4y + (3-\lambda)z &= 0 \end{aligned}\right\} \ldots\ldots\ldots(1)$$

Case (i) Take $\lambda = 0$. The system (1) becomes,

$$8x - 6y + 2z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$-6x + 7y - 4z = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

$$2x - 4y + 3z = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

Applying the rule of cross multiplication for (2) and (3) we get,

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}} \quad \therefore \frac{x}{10} = \frac{-y}{-20} = \frac{z}{20} \quad \therefore \frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

$\therefore (x, y, z)$ **are proportional to (1, 2, 2) and we can write** $x = 1, \; y = 2, \; z = 2$.

$\therefore$ **The eigen vector X_1 corresponding to the eigen value $\lambda = 0$ is X_1 = (1, 2, 2)^T**

Case (ii) Take $\lambda = 3$. The system (1) becomes,

$$5x - 6y + 2z = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$-6x + 4y - 4z = 0 \;..\ldots\ldots\ldots\ldots\ldots.(6)$$

$$2x - 4y + 0z = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots.(7)$$

Applying the rule of cross multiplication for (5) and (6) we get,

$$\frac{x}{24-8} = \frac{-y}{-20+12} = \frac{z}{20-36} \quad \therefore \frac{x}{16} = \frac{-y}{-8} = \frac{z}{-16} \quad \therefore \frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

$\therefore (x, y,\; z) =$ **(2, 1, -2)**

$\therefore$ **X_2 = (2, 1, -2)^T is the eigen vector corresponding to the eigen value $\lambda = 3$.**

Case (iii) Take $\lambda = 15$. The system (1) becomes,

$-7x - 6y + 2z = 0$…………………(8)

$-6x - 8y - 4z = 0$ ………………..(9)

$2x - 4y - 12z = 0$…………………(10)

Applying the rule of cross multiplication for (8) and (9) we get,

$\frac{x}{24+16} = \frac{-y}{28+12} = \frac{z}{56-36}$ i. e., $\frac{x}{40} = \frac{-y}{40} = \frac{z}{20}$ i. e., $\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$

$\therefore (x, y, z) =$ (2, -2, 1)

$\therefore$ $X_3 = (2, -2, 1)^T$ is the eigen vector corresponding to the eigen value $\lambda = 15$.

2. Find the eigen values and the eigen vectors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$, then the characteristic equation of A is $|A - \lambda I| = 0$.

$$\therefore \begin{vmatrix} (6-\lambda) & -2 & 2 \\ -2 & (3-\lambda) & -1 \\ 2 & -1 & (3-\lambda) \end{vmatrix} = 0$$

$\therefore (6-\lambda)[(3-\lambda)(3-\lambda) - 1] - (-2)[-2(3-\lambda) + 2] + 2[2 - 2(3-\lambda)] = 0$

$\therefore (6-\lambda)[8 - 6\lambda + \lambda^2] + 2[-4 + 2\lambda] + 2[-4 + 2\lambda] = 0$

$\therefore 48 - 36\lambda + 6\lambda^2 - 8\lambda + 6\lambda^2 - \lambda^3 + 4\lambda - 8 - 8 + 4\lambda = 0$

$\therefore -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$ $\therefore \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

By inspection, $\lambda = 2$ is a root. Now by using synthetic division method, we get,

2	1	-12	36	-32
	0	2	-20	32
	1	-10	16	0

Now we get, $\lambda^2 - 10\lambda + 16 = 0$ $\therefore (\lambda - 2)(\lambda - 8) = 0$ $\therefore \lambda = 2$ and $\lambda = 8$

$\therefore \lambda = 2, 2, 8$ are the eigen values of A.

Consider the equation $[A - \lambda I]X = 0$.

$$\left.\begin{aligned}(6-\lambda)x - 2y + 2z = 0\\ -2x + (3-\lambda)y - z = 0\\ 2x - y + (3-\lambda)z = 0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(1)$$

Case (i) Take $\lambda = 2$. The system (1) becomes,

$$4x - 2y + 2z = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$-2x + y - z = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$2x - y + z = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The above set of equations are all same as we have only one independent equation

$2x - y + z = 0$ and hence we can choose two variables arbitrarily.

Let $z = k_1$ and $y = k_2$ $\therefore 2x - k_2 + k_1 = 0$ $\therefore x = (k_2 - k_1)/2$

$\therefore X_1 = \left(\frac{k_2-k_1}{2},\ k_2,\ k_1\right)^T$ is the eigen vector corresponding to $\lambda = 2$.

where k_1, k_2 are not simultaneously equal to zero.

Case (ii) Take $\lambda = 8$. The system (1) becomes,

$$\begin{aligned}-2x - 2y + 2z = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)\\ -2x - 5y - z = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)\\ 2x - y + 5z = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)\end{aligned}$$

Applying the rule of cross multiplication for (5) and (6) we get,

$$\frac{x}{2+10} = \frac{-y}{2+4} = \frac{z}{10-4} \quad \text{i.e.,}\ \frac{x}{12} = \frac{-y}{6} = \frac{z}{6} \quad \text{i.e.,}\ \frac{x}{2} = \frac{y}{-1} = \frac{z}{1}$$

$\therefore X_2 = (2, -1, 1)^T$ is the eigen vector corresponding to $\lambda = 8$.

3. Find the eigen values and the corresponding eigen vectors of the matrix

$$\begin{bmatrix}-3 & -7 & -5\\ 2 & 4 & 3\\ 1 & 2 & 2\end{bmatrix}$$

Solution: Let $A = \begin{bmatrix}-3 & -7 & -5\\ 2 & 4 & 3\\ 1 & 2 & 2\end{bmatrix}$, then the characteristic equation of A is $|A - \lambda I| = 0$.

$$\therefore \begin{vmatrix}(-3-\lambda) & -7 & -5\\ 2 & (4-\lambda) & 3\\ 1 & 2 & (2-\lambda)\end{vmatrix} = 0$$

$\therefore (-3-\lambda)[(4-\lambda)(2-\lambda)-6]-(-7)[2(2-\lambda)-3]+(-5)[4-1(4-\lambda)]=0$

$\therefore\ (-3-\lambda)[2-6\lambda+\lambda^2]+7[1-2\lambda]-5[\lambda]=0$

$\therefore -6+18\lambda-3\lambda^2-\ 2\lambda+6\lambda^2-\lambda^3+7-14\lambda-5\lambda=0$

$\therefore -\lambda^3+3\lambda^2-3\lambda+1=0 \quad \therefore \lambda^3-3\lambda^2+3\lambda-1=0$

$\therefore\ (\lambda-1)^3=0\ \therefore \lambda\ =1,\ 1,\ 1$ **are the eigen values of A.**

Consider the equation $[A-\lambda I]X=0$

$(-3-\lambda)x-7y-5z\ =0$

$2x+(4-\lambda)y+3z=0$

$x+2y+(2-\lambda)z=0$

Take $\lambda=1$, **then we get,**

$-4x\ -7y-5z=0 \ldots\ldots\ldots\ldots(1)$
$2x+3y+3z=0 \ldots\ldots\ldots\ldots(2)$
$1x+2y+1z=0 \ldots\ldots\ldots\ldots(3)$

Applying the rule of cross multiplication for (1) and (2) we get,

$\frac{x}{-21+15}=\frac{-y}{-12+10}=\frac{z}{-12+14}$ i.e., $\frac{x}{-6}=\frac{-y}{-2}=\frac{z}{2}$ i.e., $\frac{x}{3}=\frac{y}{-1}=\frac{z}{-1}$

$\therefore$ **X = (3, -1, -1)^T is the eigen vector corresponding to λ = 1.**

HOME WORK:

1. Find the Eigen values and corresponding eigen vectors of the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

2. Find the Eigen values and corresponding eigen vectors of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

3. Find the Eigen values and corresponding eigen vectors of the matrix $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

4. Find the Eigen values and corresponding eigen vectors of the matrix
$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Rayleigh's power method:

Rayleigh's power method is an iterative method used to determine the numerically largest eigen value (dominant eigen value) and corresponding eigen vector of a square matrix.

Working procedure:

(i) Suppose A is the given square matrix, then we assume that initially $X^{(0)}$ as an eigen vector corresponding to the largest eigen value λ.

For example we may assume that $X^{(0)}$ as $[1, 0, 0]'$ or $[0, 0, 1]'$ or $[1, 1, 1]'$

(ii) Find the matrix product $A X^{(0)}$ which gives a column matrix.

(iii) Take out the numerically largest element as the common factor to obtain

$A X^{(0)} = \lambda^{(1)} X^{(1)}$. Where $\lambda^{(1)}$ is first approximation to the largest eigen value and $X^{(1)}$ is the corresponding eigen vector.

Similarly, find $AX^{(1)} = \lambda^{(2)} X^{(2)}$, $AX^{(2)} = \lambda^{(3)} X^{(3)}$, $AX^{(3)} = \lambda^{(4)} X^{(4)}$………..

(iv) Continue this process until to obtain the values of λ and X up to a desired degree of accuracy

Problems:

1. Find the Largest Eigen value and the corresponding Eigen vector of the matrix

$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ by the power method. Perform five iteration. Take $[1, 0, 0]^T$ as initial approximation.

Solution: **Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$**

Let the initial approximation to the required Eigen vector $X^{(0)} = [1, 0, 0]^T$

$$AX^{(0)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 0.17 \\ 0.67 \\ 1 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.17 \\ 0.67 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.16 \\ 2.36 \\ 8.03 \end{bmatrix} = 8.03 \begin{bmatrix} 0.02 \\ 0.29 \\ 1 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.02 \\ 0.29 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.15 \\ 0.24 \\ 5.99 \end{bmatrix} = 5.99 \begin{bmatrix} 0.19 \\ 0.04 \\ 1 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.19 \\ 0.04 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.07 \\ -0.08 \\ 6.26 \end{bmatrix} = 6.26 \begin{bmatrix} 0.33 \\ -0.01 \\ 1 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.32 \\ -0.01 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.35 \\ 0.24 \\ 6.89 \end{bmatrix} = 6.89 \begin{bmatrix} 0.34 \\ 0.03 \\ 1 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

Thus after 5 iterations the numerically largest Eigen value is 6.89 and corresponding eigen vector is $[0.34, 0.03, 1]^T$

2. Find the largest eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ by using power method.

Solution: Let $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Let the initial approximation to the required eigen vector $X^{(0)} = [1, 0, 0]^T$. Then

$$AX^{(0)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.8 \\ 1.2 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \begin{bmatrix} 3 \\ -3.43 \\ 1.86 \end{bmatrix} = 3.43 \begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \begin{bmatrix} 2.74 \\ -3.41 \\ 2.08 \end{bmatrix} = 3.41 \begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \begin{bmatrix} 2.60 \\ -3.41 \\ 2.22 \end{bmatrix} = 3.41\begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \lambda^{(6)}X^{(6)}$$

Hence the largest eigen value is 3.41 and the corresponding eigen vector is

$[0.76, \ -1, \ 0.65]'$.

3. Find the largest eigen value and the corresponding eigen vector of the matrix

$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **using the power method by taking the initial approximation to the eigen vector as [1 ,1,1]^T.**

Solution: **Let A =** $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Let the initial approximation to the required Eigen vector $X^{(0)} = [1, 1, 1]^T$

$$AX^{(0)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} = \lambda^{(1)}X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} = \begin{bmatrix} 7.34 \\ -2.67 \\ 4.01 \end{bmatrix} = 7.34\begin{bmatrix} 1 \\ -0.36 \\ 0.55 \end{bmatrix} = \lambda^{(2)}X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.36 \\ 0.55 \end{bmatrix} = \begin{bmatrix} 7.82 \\ -3.63 \\ 4.01 \end{bmatrix} = 7.82\begin{bmatrix} 1 \\ -0.46 \\ 0.51 \end{bmatrix} = \lambda^{(3)}X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.46 \\ 0.51 \end{bmatrix} = \begin{bmatrix} 7.94 \\ -3.89 \\ 3.99 \end{bmatrix} = 7.94\begin{bmatrix} 1 \\ -0.49 \\ 0.5 \end{bmatrix} = \lambda^{(4)}X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.49 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 7.98 \\ -3.97 \\ 3.99 \end{bmatrix} = 7.98\begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \lambda^{(5)}X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 4 \end{bmatrix} = 8\begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \lambda^{(6)}X^{(6)}$$

Hence the largest Eigen value is 8 and the corresponding Eigen vector is [1, - 0.5 ,0.5]^T

4. Find the largest eigen value and the corresponding eigen vector of the matrix

$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ using the power method by taking the initial approximation to the eigen vector as [1, 0.8 , - 0.8]^T. Perform five iterations.

Solution: Let A = $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$

Given the initial approximation to the required eigen vector is $X^{(0)}$ =[1, 0.8, - 0.8]^T.

Then

$$AX^{(0)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 5.2 \\ -5.2 \end{bmatrix} = 5.6\begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \lambda^{(1)}X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \begin{bmatrix} 5.86 \\ 5.72 \\ -5.72 \end{bmatrix} = 5.86\begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \lambda^{(2)}X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \begin{bmatrix} 5.96 \\ 5.92 \\ -5.92 \end{bmatrix} = 5.96\begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \lambda^{(3)}X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \begin{bmatrix} 5.98 \\ 5.96 \\ -5.96 \end{bmatrix} = 5.98\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \lambda^{(4)}X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ -6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$AX^{(5)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ -6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Thus after 5 iterations the numerically largest eigen value is 6 and corresponding eigen vector is [1, 1, -1]^T

HOME WORK:

1. Determine the largest (dominant) eigen value and the corresponding eigen vector of the

matrix $\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$ **using the power method. Take [1,0,0]' as initial approximation.**

2. Find the Largest Eigen value and the corresponding Eigen vector of the matrix

A = $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ **using the power method by taking the initial Eigen vector as** $[1, 0, 0]'$.

3. Determine the largest (dominant) eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **using the power method. Taking the initial Eigen vector as** $[1, 0, 0]'$.

4. Determine the largest (dominant) eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ **using the power method, taking initial approximation** $(1, 1, 0)^T$.

5. Find the largest Eigen value and the corresponding Eigen vector by power method given that $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ **carry out 5 iterations. (Use** $[1\ 0\ 0]^T$ **as the initial vector).**