NAGARJUNA COLLEGE OF ENGINEERING AND TECHNOLOGY

(An autonomous institution under VTU)

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NAGARJUNA
COLLEGE OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

CALCULUS AND LINEAR ALGEBRA

(COURSE CODE 23MATS11)

MODULE-4

INTEGRAL CALCULUS

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MODULE-4

INTEGRAL CALCULUS

SYLLABUS:

Multiple Integrals: Evaluation of double and triple integrals, evaluation of double integrals by change of order of integration, changing into polar coordinates.

Beta and Gamma functions: Definitions, properties, the relation between Beta and Gamma functions,Problems.

Multiple Integrals:

Double integrals:

Let f(x, y) be a function of two independent variable x and y defined at each point in the finite region R of the xy-plane, then the integral of f(x, y) over the region R is written as $\iint_R f(x, y)\, dA$ and is called the double integral.

If x varies from a value x_1 to x_2 and y varies from a value y_1 to y_2 in the region R then we write $\iint_R f(x, y)\, dA = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\, dy\, dx$, where x_1, x_2, y_1, y_2 are constants.

If x_1 and x_2 are constants, $y_1 = f_1(x)$ and $y_2 = f_2(x)$ then

$$\iint_R f(x, y)\, dA = \int_{x_1}^{x_2} \left[\int_{y_1}^{y_2} f(x, y)\, dy\right] dx.$$

If y_1 and y_2 are constants, $x_1 = f_1(y)$ and $x_2 = f_2(y)$ then

$$\iint_R f(x, y)\, dA = \int_{y_1}^{y_2} \left[\int_{x_1}^{x_2} f(x, y)\, dx\right] dy$$

Problems:

1. Evaluate $\int_1^2 \int_1^3 xy^2\, dx\, dy$

Solution:

Let $I = \int_1^2 \int_1^3 xy^2\, dx\, dy = \int_1^2 y^2 \left[\int_1^3 x dx\right] dy = \int_1^2 y^2 \left[\frac{x^2}{2}\right]_1^3 dy$

$\therefore\ I = \int_1^2 \frac{y^2}{2}(9 - 1)\, dy = \int_1^2 4y^2 dy = 4\left[\frac{y^3}{3}\right]_1^2 = \frac{4}{3}(8 - 1) = \frac{28}{3}$

2. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$

Solution:

Let $I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} xy\, dy\, dx = \int_{x=0}^{1} x \left[\int_{y=x}^{\sqrt{x}} y\, dy\right] dx = \int_{x=0}^{1} x \left[\frac{y^2}{2}\right]_x^{\sqrt{x}} dx$

$= \int_{x=0}^{1} \frac{x}{2} \left[(\sqrt{x})^2 - x^2\right] dx = \frac{1}{2}\int_0^1 x(x - x^2) dx = \frac{1}{2}\int_0^1 (x^2 - x^3) dx$

$\therefore\ I = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}\left[\left(\frac{1}{3} - \frac{1}{4}\right) - (0 - 0)\right] = \frac{1}{24}$

3. Evaluate $\int_0^5 \int_0^{x^2} x(x^2 + y^2)\, dx\, dy$

Solution:

Let $I = \int_0^5 \int_0^{x^2} x(x^2 + y^2)\, dx\, dy = \int_0^5 \left[\int_0^{x^2} (x^3 + xy^2)\, dy\right] dx = \int_{x=0}^{x=5} \left[x^3 y + x\frac{y^3}{3}\right]_0^{x^2} dx$

$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx = \left(\frac{x^6}{6} + \frac{x^8}{24}\right)_0^5 = \frac{1}{6}(5^6) + \frac{1}{24}(5^8) = \frac{15625}{6} + \frac{390625}{24}$

$\therefore\ I = \frac{62500+390625}{24} = \frac{453125}{24} \cong 18880.2$

4. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{\sqrt{1+x^2+y^2}}$

Solution:

Let $I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{\sqrt{1+x^2+y^2}}$ For convenience take $1 + x^2 = a^2$.

$\therefore\ I = \int_0^1 \int_0^a \frac{dy\, dx}{\sqrt{a^2+y^2}} = \int_{x=0}^{x=1} \left[\int_{y=0}^{y=a} \frac{dy}{\sqrt{a^2+y^2}}\right] dx$

$= \int_{x=0}^{x=1} \left[\log(y + \sqrt{a^2 + y^2})\right]_0^a dx.$ Using $\int \frac{dx}{\sqrt{a^2+x^2}} = \log(x + \sqrt{a^2 + x^2})$

$\therefore\ I = \int_{x=0}^{x=1} \left[\log(a + \sqrt{a^2 + a^2}) - \log(0 + \sqrt{a^2 + 0})\right] dx$

$= \int_{x=0}^{x=1} \left[\log(a + \sqrt{2a^2}) - \log(\sqrt{a^2})\right] dx = \int_{x=0}^{x=1} \left[\log(a(1 + \sqrt{2})) - \log a\right] dx$

$= \int_{x=0}^{x=1} \left[\log a + \log(1 + \sqrt{2}) - \log a\right] dx = \int_{x=0}^{x=1} \left[\log(1 + \sqrt{2})\right] dx$

$\therefore\ I = \log(1 + \sqrt{2}) \int_{x=0}^{x=1} dx = \log(1 + \sqrt{2})\, [x]_0^1 = \log(1 + \sqrt{2})(1 - 0) = \log(1 + \sqrt{2})$

5. Evaluate $\int_0^{\pi/2}\int_0^{a\cos\theta}\frac{a\,r}{\sqrt{a^2-r^2}}\ dr\,d\theta$

Solution:

Let $I=\int_0^{\pi/2}\int_0^{a\cos\theta}\frac{a\,r}{\sqrt{a^2-r^2}}\ dr\,d\theta$ Take substitution $a^2-r^2=t$.

Differentiating we get, $-2r\,dr=dt\ \therefore\ r\,dr=-\frac{dt}{2}$; when $r=0,\ t=a^2$ and when $r=a\cos\theta,\ t=a^2-a^2\cos^2\theta=a^2(1-\cos^2\theta)=a^2\sin^2\theta$.

$$\therefore\ I=\int_0^{\frac{\pi}{2}}\left\{\int_{a^2}^{a^2\sin^2\theta}\frac{a}{\sqrt{t}}\cdot\left(-\frac{dt}{2}\right)\right\}d\theta=-\frac{a}{2}\int_0^{\frac{\pi}{2}}\left[\frac{t^{1/2}}{1/2}\right]_{a^2}^{a^2\sin^2\theta}d\theta$$

$$=-a\int_0^{\frac{\pi}{2}}(a\sin\theta-a)\,d\theta$$

$$=-a^2\int_0^{\frac{\pi}{2}}(\sin\theta-1)\,d\theta=-a^2[-\cos\theta-\theta]_0^{\frac{\pi}{2}}=a^2[\cos\theta+\theta]_0^{\frac{\pi}{2}}$$

$$\therefore\ I=a^2\left[(0-1)+\left(\frac{\pi}{2}-0\right)\right]=a^2\left[\frac{\pi}{2}-1\right]$$

6. Evaluate $\int_0^{\pi/2}\int_0^{a\sin\theta}r^3\sin^2\theta\ dr\,d\theta$

Solution:

Let $I=\int_0^{\pi/2}\int_0^{a\sin\theta}r^3\sin^2\theta\ dr\,d\theta\ =\int_0^{\frac{\pi}{2}}\sin^2\theta\left[\frac{r^4}{4}\right]_0^{a\sin\theta}d\theta$

$$\therefore\ I=\frac{1}{4}\int_0^{\frac{\pi}{2}}\sin^2\theta\,(a^4\sin^4\theta-0)\,d\theta=\frac{a^4}{4}\int_0^{\frac{\pi}{2}}\sin^6\theta\ d\theta$$

Use Reduction Formula, $\int_0^{\frac{\pi}{2}}\sin^n\theta\ d\theta=\frac{(n-1)}{n}\frac{(n-3)}{(n-2)}\frac{(n-5)}{(n-4)}$..........K,

where K = 1 if n is odd and K = $\frac{\pi}{2}$ if n is even.

Here n = 6 (even)

$$\therefore\ I=\frac{a^4}{4}\frac{(6-1)}{6}\frac{(6-3)}{(6-2)}\frac{(6-5)}{(6-4)}\frac{\pi}{2}=\frac{a^4}{4}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}=\frac{5a^4\pi}{128}.$$

HOME WORK:

1. Evaluate: $\int_0^3\int_1^2 xy(1+x+y)\,dy\,dx$

2. Evaluate: $\int_1^2\int_3^4(xy+e^y)\ dx\,dy$

3. *Evaluate* $\int_0^1\int_0^1\frac{dy\,dx}{\sqrt{1-x^2}\sqrt{1-y^2}}$

4. *Evaluate* $\int_0^1\int_0^{\sqrt{1-y^2}}x^3y\,dx\,dy$

5. Evaluate: $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dx\, dy$

6. Evaluate: $\int_0^1 \int_0^y e^{x/y}\, dx\, dy$

7. Evaluate: $\int_0^1 \int_0^x e^{y/x}\, dy\, dx$

8. Evaluate: $\int_1^2 \int_0^x \frac{dy\, dx}{x^2+y^2}$

9. Evaluate: $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2}$

10. *Evaluate* $\int_0^1 \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\, dx\, dy$

Triple integrals:

If $f(x, y, z)$ is a function of three independent variables defined in a finite region V, then the

integration of $f(x, y, z)$ over the region V is denoted by

$$\iiint_V f(x, y, z) dV = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z)\, dz\, dy\, dx$$

Problems:

1. **Evaluate $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2) dz\, dy\, dx$.**

Solution:

Let $I = \int_{x=-c}^{c} \int_{y=-b}^{b} \int_{z=-a}^{a} (x^2 + y^2 + z^2)\, dz\, dy\, dx$

$\therefore\ I = \int_{x=-c}^{c} \int_{y=-b}^{b} \left[x^2 z + y^2 z + \frac{z^3}{3}\right]_{-a}^{a} dy\, dx$

$= \int_{x=-c}^{c} \int_{y=-b}^{b} [x^2(a + a) + y^2(a + a) + (\frac{a^3}{3} + \frac{a^3}{3})]\, dy\, dx$

$= \int_{x=-c}^{c} \int_{y=-b}^{b} [2ax^2 + 2ay^2 + \frac{2a^3}{3}]\, dy\, dx = \int_{x=-c}^{c} \left[2ax^2 y + 2a\left(\frac{y^3}{3}\right) + \left(\frac{2a^3}{3}\right) y\right]_{-b}^{b} dx$

$= \int_{x=-c}^{c} \left[2ax^2(b + b) + \frac{2a}{3}(b^3 + b^3) + \frac{2a^3}{3}(b + b)\right] dx$

$= \int_{x=-c}^{c} \left[4abx^2 + \frac{4ab^3}{3} + \frac{4a^3 b}{3}\right] dx = \left[(4ab)\left(\frac{x^3}{3}\right) + \frac{4ab^3}{3}.x + \frac{4a^3 b}{3}.x\right]_{-c}^{c}$

$= \left[(4ab)\left(\frac{c^3}{3} + \frac{c^3}{3}\right) + \frac{4ab^3}{3}.(c + c) + \frac{4a^3 b}{3}.(c + c)\right]$

$= \frac{8abc^3}{3} + \frac{8ab^3 c}{3} + \frac{8a^3 bc}{3}$

$\therefore\ I = \frac{8abc(a^2+b^2+c^2)}{3}$.

2. Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dx\,dy\,dz$

Solution:

Let $I = \int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z}(x+y+z)dy\,dx\,dz$

$\therefore\ I = \int_{z=-1}^{1}\int_{x=0}^{z}\left[xy+\frac{y^2}{2}+zy\right]_{x-z}^{x+z}dx\,dz$

$= \int_{z=-1}^{1}\int_{x=0}^{z}\left[x\{(x+z)-(x-z)\}+\frac{1}{2}\{(x+z)^2-(x-z)^2\}+z\{(x+z)-(x-z)\}\right]dx\,dz$

$= \int_{z=-1}^{1}\int_{x=0}^{z}\left[x\{2z\}+\frac{1}{2}\{(x^2+z^2+2xz)-(x^2+z^2-2xz)\}+z\{2z\}\right]dx\,dz$

$= \int_{z=-1}^{1}\int_{x=0}^{z}[2xz+2xz+2z^2]dx\,dz = \int_{z=-1}^{1}\int_{x=0}^{z}[4xz+2z^2]dx\,dz$

$= \int_{z=-1}^{1}\left[4z\,\frac{x^2}{2}+2z^2x\right]_0^z dz = \int_{z=-1}^{1}[2z\,(z^2-0)+2z^2(z-0)]$

$\therefore\ I = \int_{z=-1}^{1}(2z^3+2z^3)dz = \int_{z=-1}^{1}4z^3dz = \left[4\,\frac{z^4}{4}\right]_{-1}^{1} = 1-1 = 0$

3. Evaluate $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}}xyz\,dx\,dy\,dz$

Solution:

Let $I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}}xyz\,dz\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}xy\left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}}dy\,dx$

$\therefore\ I = \frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}xy\,(1-x^2-y^2)\,dy\,dx = \frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}(xy-x^3y-xy^3)\,dy\,dx$

$\therefore\ I = \frac{1}{2}\int_{x=0}^{1}\left[x\frac{y^2}{2}-x^3\frac{y^2}{2}-x\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}}dx = \frac{1}{2}\int_{x=0}^{1}\frac{1}{4}[2xy^2-2x^3y^2-xy^4]_0^{\sqrt{1-x^2}}dx$

$\therefore\ I = \frac{1}{8}\int_{x=0}^{1}[2x\,(1-x^2)-2x^3(1-x^2)-x\,(1-x^2)^2]\,dx$

$\therefore\ I = \frac{1}{8}\int_{x=0}^{1}[2x\,(1-x^2)-2x^3(1-x^2)-x(1-2x^2+x^4)]\,dx$

$\therefore\ I = \frac{1}{8}\int_{x=0}^{1}[2x - 2x^3 - 2x^3 + 2x^5 - x + 2x^3 - x^5]\,dx$

$\therefore\ I = \frac{1}{8}\int_{x=0}^{1}[x - 2x^3 + x^5]\,dx = \frac{1}{8}\left[\frac{x^2}{2} - 2\frac{x^4}{4} + \frac{x^6}{6}\right]_0^1 = \frac{1}{8}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right]_0^1 = \frac{1}{8}.\frac{1}{6} = \frac{1}{48}$

4. Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz$

Solution:

Let $I = \int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}dx\,dy\,dz = \int_{x=0}^{\log 2}\int_{y=0}^{x}\int_{z=0}^{x+\log y} e^x e^y[e^z dz]dy\,dx$

$\therefore\ I = \int_{x=0}^{\log 2}\int_{y=0}^{x} e^x e^y[e^z]_0^{x+\log y}\,dy\,dx = \int_{x=0}^{\log 2}\int_{y=0}^{x} e^x e^y\left[e^{x+\log y} - e^o\right]dy\,dx$

$\therefore\ I = \int_{x=0}^{\log 2}\int_{y=0}^{x} e^x e^y\left[e^x e^{\log y} - 1\right]dy\,dx = \int_{x=0}^{\log 2}\int_{y=0}^{x} e^x e^y[e^x y - 1]dy\,dx$

$\therefore\ I = \int_{x=0}^{\log 2}\int_{y=0}^{x}[e^{2x}\,y\,e^y - e^x e^y]\,dy\,dx = \int_{x=0}^{\log 2}[e^{2x}(ye^y - e^y) - e^x e^y]_0^x\,dx$

$\therefore\ I = \int_{x=0}^{\log 2}[e^{2x}\{(xe^x - e^x) - (0-1)\} - e^x(e^x - 1)]\,dx$

$\therefore I = \int_{x=0}^{\log 2}[xe^{3x} - e^{3x} + e^{2x} - e^{2x} + e^x]\,dx = \int_{x=0}^{\log 2}[xe^{3x} - e^{3x} + e^x]\,dx$

$\therefore\ I = \left[\left(x\frac{e^{3x}}{3} - \frac{e^{3x}}{9}\right) - \frac{e^{3x}}{3} + e^x\right]_0^{\log 2} = \left[x\frac{e^{3x}}{3} - \frac{4e^{3x}}{9} + e^x\right]_0^{\log 2}$

$\therefore I = [(\log 2)\frac{e^{3\log 2}}{3} - 0] - [\frac{4e^{3\log 2}}{9} - \frac{4e^0}{9}] + [e^{\log 2} - e^0]$

$\therefore I = (\log 2)\,\frac{2^3}{3} - \frac{4(2^3)}{9} + \frac{4}{9} + 2 - 1$

$\therefore I = \frac{8\log 2}{3} - \frac{32}{9} + \frac{4}{9} + 1$

$\therefore\ I = \frac{8\log 2}{3} - \frac{28}{9} + 1 = \frac{\log 256}{3} - \frac{19}{9}$

5. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$

Solution:

Let $I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$.

For convenience, we take, $k^2 = 1 - x^2 - y^2$

$$\therefore\ I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{k^2-z^2}}\, dy\, dx = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{0}^{k} \frac{dz}{\sqrt{k^2-z^2}}\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left[\sin^{-1}\frac{z}{k}\right]_{z=0}^{k} dy\, dx = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} [\sin^{-1} 1 - \sin^{-1} 0]\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left(\frac{\pi}{2} - 0\right) dy\, dx = \frac{\pi}{2} \int_{x=0}^{1} [y]_0^{\sqrt{1-x^2}}\, dx = \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right]_{x=0}^{1} \quad \text{Using } \int \sqrt{a^2-x^2}\ dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore\ I = \frac{\pi}{2}\left[(0-0) + \frac{1}{2}(\sin^{-1}1 - \sin^{-1}0)\right] = \frac{\pi}{2}.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi^2}{8}$$

6. Evaluate: $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{\frac{a^2-r^2}{a}} r\, dz\, dr\, d\theta$.

Solution:

Let $I = \int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{\frac{a^2-r^2}{a}} r\, dz\, dr\, d\theta = \int_0^{\pi/2} \int_0^{a\sin\theta} r[z]_0^{\frac{a^2-r^2}{a}}\, dr\, d\theta$

$$= \int_0^{\pi/2} \int_0^{a\sin\theta} r\left[\frac{a^2-r^2}{a}\right] dr\, d\theta = \frac{1}{a}\int_0^{\pi/2} \int_0^{a\sin\theta} [a^2 r - r^3]\ dr\, d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2} \left[a^2\frac{r^2}{2} - \frac{r^4}{4}\right]_0^{a\sin\theta} d\theta = \frac{1}{a}\int_0^{\pi/2} \left[a^2\frac{a^2\sin^2\theta}{2} - \frac{a^4\sin^4\theta}{4}\right] d\theta$$

$$= \frac{a^3}{2}\int_0^{\pi/2} \sin^2\theta\, d\theta - \frac{a^3}{4}\int_0^{\pi/2} \sin^4\theta\, d\theta$$

Use Reduction Formula, $\int_0^{\frac{\pi}{2}} \sin^n\theta\ d\theta = \frac{(n-1)}{n}\frac{(n-3)}{(n-2)}\frac{(n-5)}{(n-4)}$**..........K,**

where K = 1 if n is odd and K = $\frac{\pi}{2}$ **if n is even.**

$$\therefore\ I = \frac{a^3}{2}.\frac{1}{2}.\frac{\pi}{2} - \frac{a^3}{4}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi a^3}{8} - \frac{3\pi a^3}{64} = \frac{8\pi a^3 - 3\pi a^3}{64} = \frac{5\pi a^3}{64}.$$

HOME WORK:

1. Evaluate: $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$.

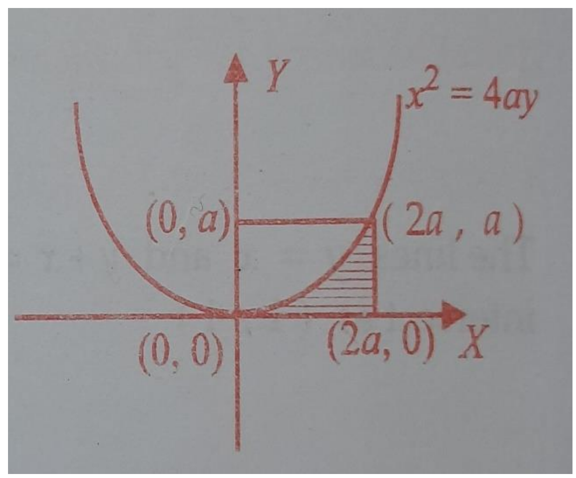
2. Evaluate: $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dz\, dx\, dy$.

3. Evaluate: $\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2) dx\, dy\, dz.$

Evaluation of double integral in a given region.

1. *Evaluate* $\iint_A$ xy $dx\, dy$ Where A is the domain bounded by ordinate $x = 2a$ and the curve $x^2 = 4ay$

Solution:



Here x varies from $x = 0$ to $x = 2a$ and

y varies from y = 0 to $y = \frac{x^2}{4a}$.

$\therefore \quad I = \iint_A xy\ dx\, dy = \int_{x=0}^{2a} \int_{y=0}^{\frac{x^2}{4a}} xy\ dydx$

$= \int_{x=0}^{2a} x \left[\frac{y^2}{2}\right]_0^{\frac{x^2}{4a}} dx = \int_0^{2a} \frac{x}{2}\left[\frac{x^4}{16a^2} - 0\right] dx$

$= \frac{1}{32a^2}\int_0^{2a} x^5\, dx = \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$

$\therefore \quad I = \frac{1}{32a^2}\left[\frac{2^6a^6}{6} - 0\right] = \frac{a^4}{3}$.

Otherwise,

The point of intersection of the line $x = 2a$ and the parabola $x^2 = 4ay$ is $(2a,\ a)$.

$\therefore$ x varies from $x = 2\sqrt{ay}$ to $x = 2a$ and y varies from y = 0 to y = a.

$\therefore \quad I = \iint_A xy\ dx\, dy = \int_{y=0}^{a} \int_{x=2\sqrt{ay}}^{2a} xy\ dxdy = \int_{y=0}^{a} y\left[\frac{x^2}{2}\right]_{2\sqrt{ay}}^{2a} dy = \int_0^a \frac{y}{2}[4a^2 - 4ay]\, dy$

$= \int_0^a [2a^2y - 2ay^2]\, dy = \left[2a^2\frac{y^2}{2} - 2a\frac{y^3}{3}\right]_0^a = \left[a^4 - \frac{2a^4}{3}\right] = \frac{a^4}{3}$.

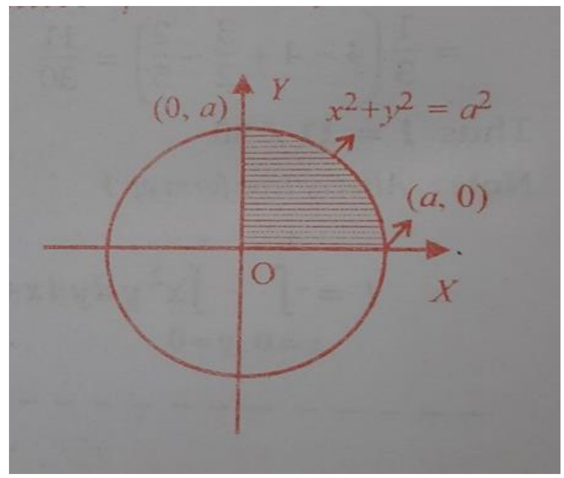
2. ***Evaluate*** $\iint xy\,dx\,dy$ ***over the positive quadrant of the circle*** $x^2+y^2=a^2$

Solution:

Given $x^2+y^2=a^2 \;\therefore\; y^2=a^2-x^2 \;\therefore\; y=\sqrt{a^2-x^2}$

∴ x varies from x = 0 to x = a **and y varies from y = 0 to y =** $\sqrt{a^2-x^2}$ **in the first quadrant.**

$$\therefore\; I=\iint xy\,dx\,dy=\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\,dx$$



$$\therefore\; I=\int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$\therefore\; I=\frac{1}{2}\int_{x=0}^{a} x(a^2-x^2)dx=\frac{1}{2}\int_{x=0}^{a}(a^2x-x^3)dx$$

$$=\frac{1}{2}\left[a^2\frac{x^2}{2}-\frac{x^4}{4}\right]_{x=0}^{a}=\frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right]=\frac{a^4}{8}$$

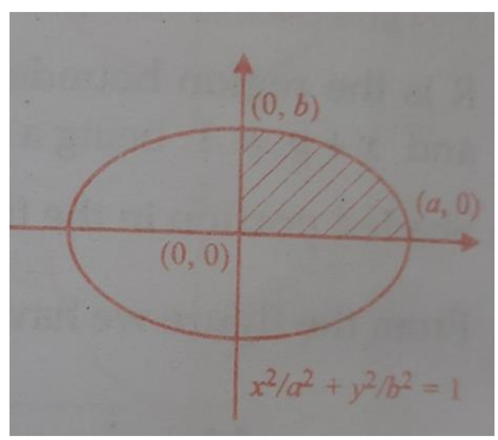
$$\therefore\; I=\frac{a^4}{8}.$$

3. ***Evaluate*** $\iint y\,dx\,dy$ ***over the region by the first quadrant of the ellipse*** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Solution:

Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \;\therefore\; \frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{(a^2-x^2)}{a^2}$

$$\therefore\; y^2=\frac{b^2(a^2-x^2)}{a^2}$$



∴ x varies from x = 0 to x = a **and y varies from y = 0 to y =** $\frac{b}{a}\sqrt{a^2-x^2}$ **in the first quadrant.**

$$\therefore\; \mathrm{I}=\iint y\,dx\,dy=\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} y\,dy\,dx$$

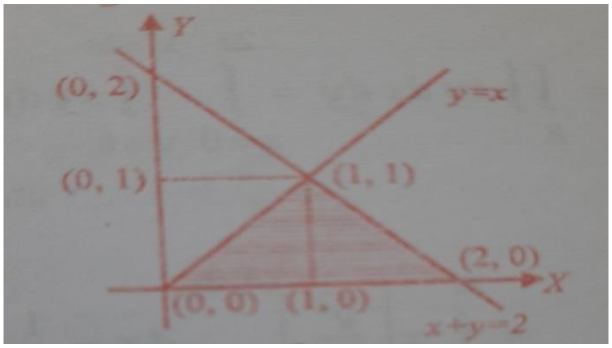
$$=\int_{x=0}^{a}\left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx=\int_{x=0}^{a}\frac{b^2}{2a^2}(a^2-x^2)\,dx$$

$\therefore\quad I=\frac{b^2}{2a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a=\frac{b^2}{2a^2}\left[\left(a^3-\frac{a^3}{3}\right)-0\right]=\frac{b^2}{2a^2}\left(\frac{2a^3}{3}\right)\ =\frac{ab^2}{3}$

4. ***Evaluate*** $\iint_R x^2y\ dx\ dy$ ***where R is the region bounded by the lines.***

$y=x\ ,\ x+y=2$ ***and*** $y=0$

Solution:



Solving the equations $y=x$ ***and*** $x+y=2$ **we get x = 1 and y = 1**

$\therefore$ **The lines** $y=x,\ \ x+y=2$ ***intersect at*** $(1,1)$.

$\therefore$ **y varies from y = 0 to y = 1 and**

x varies from x = y to x = 2 $-y$ **in the first quadrant.**

$\therefore\ I=\iint x^2y\,dx\,dy\ \ =\int_{y=0}^{1}\int_{x=y}^{2-y}x^2y\,dx\,dy$

$=\int_{y=0}^{1}\left[\frac{x^3}{3}\right]_{x=y}^{2-y}dy=\frac{1}{3}\int_{y=0}^{1}y\{(2-y)^3-y^3\}\,dy=\frac{1}{3}\int_{y=0}^{1}y(8-12y+6y^2-y^3-y^3)dy$

$\therefore\ I=\frac{1}{3}\int_{y=0}^{1}(8y-12y^2+6y^3-2y^4)\,dy=\frac{1}{3}\left[4y^2-4y^3+\frac{3}{2}y^4-2\frac{y^5}{5}\right]_{y=0}^{1}$

$\therefore\ I=\frac{1}{3}\left(4-4+\frac{3}{2}-\frac{2}{5}\right)=\frac{1}{3}\left(\frac{11}{10}\right)=\frac{11}{30}$

Alternatively, we can take,

I = $\iint_R x^2\,y\,dx\,dy=\int_{x=0}^{x=1}\int_{y=0}^{y=x}x^2y\,dydx+\int_{x=1}^{x=2}\int_{y=0}^{y=2-x}x^2y\,dydx=\frac{11}{30}$

HOME WORK:

1. ***Evaluate*** $\iint_R$ **y** $dx\ dy$ **Where R is the region bounded by the parabolas**

$y^2=4x$ **and** $x^2=4y.$

2. ***Evaluate*** $\iint_R$ **x y (x + y)** $dx\ dy$ **Where R is the region bounded by the parabola**

y = x^2 **and the line y = x.**

Change of order of integration:

In a double integral with variable limits, the change of order of integration changes the limit of

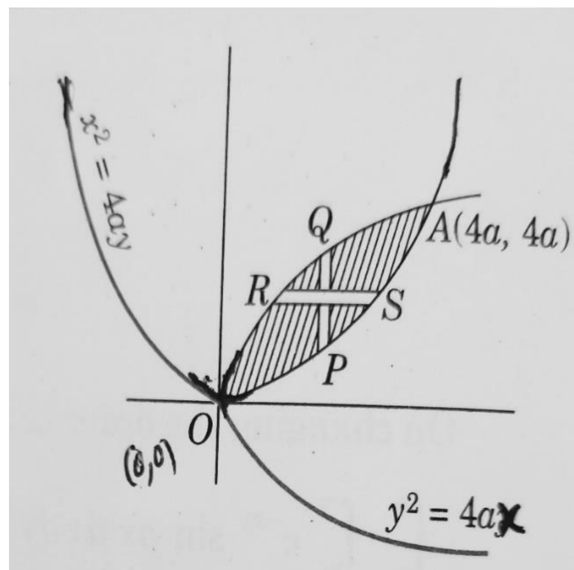
integration.

i.e., $\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dy\,dx = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy$

Problems:

1. Evaluate $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$ by changing the order of integration.

Solution:



Given $I = \int_{x=0}^{x=4a}\int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dy\,dx$……….(1)

Here x varies from x = 0 to x = 4a and

y varies from $y = \frac{x^2}{4a}$ **and** $y = 2\sqrt{ax}$ **.**

∴ The region of the integration is the region

bounded by the parabolas $x^2 = 4ay$ **and** $y^2 = 4ax$**.**

These two parabolas are intersecting

at (0, 0) and (4a, 4a)

∴ By changing the order of integration, x varies

from $x = \frac{y^2}{4a}$ **to** $x = 2\sqrt{ay}$ **and y varies from**

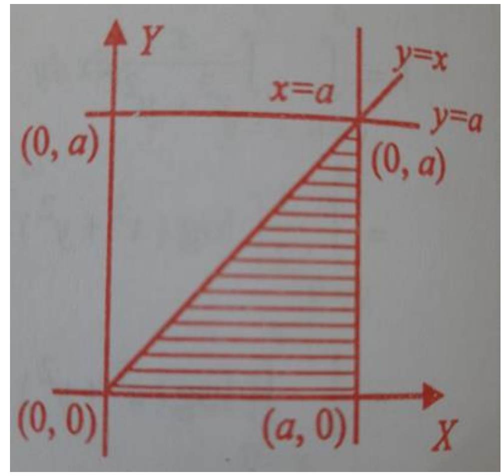
$y = 0$ **and** $y = 4a$.

$$\therefore\ I = \int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a}[x]_{\frac{y^2}{4a}}^{2\sqrt{ay}}\,dy = \int_{y=0}^{4a}\left[2\sqrt{ay} - \frac{y^2}{4a}\right]dy$$

$$\therefore\ I = \left\{\frac{2\sqrt{a}y^{3/2}}{3/2} - \frac{y^3}{12a}\right\}_0^{4a} = \frac{4.a^{1/2}.4^{3/2}.a^{3/2}}{3} - \frac{64a^3}{12a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

2. Evaluate $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ by changing the order of integration.

Solution:



Given $I = \int_{y=0}^{a} \int_{x=y}^{a} \frac{x\,dx\,dy}{x^2+y^2}$……..(1)

Here x varies from x = y to x = a and y varies from $y = 0$ and $y = a$.

∴ The region of the integration is the region bounded by the lines y = 0, y = x and x = a.

By changing the order of integration, x varies from $x = 0$ to x = a and y varies from $y = 0$ and $y = x$.

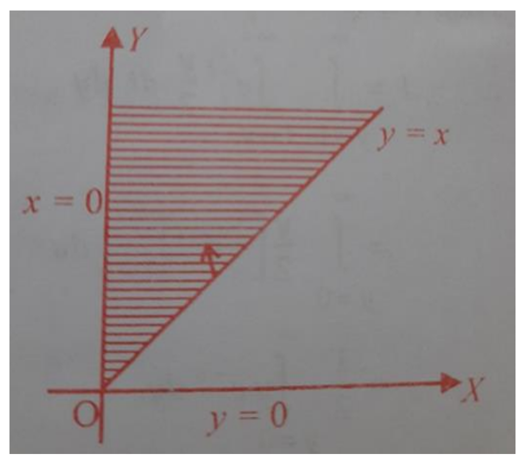
$$\therefore \quad I = \int_{x=0}^{a} \int_{y=0}^{x} x.\frac{1}{x^2+y^2}\,dy\,dx$$

$$= \int_{x=0}^{a} x.\frac{1}{x}\left[tan^{-1}\left({}^{y}/_{x}\right)\right]_{y=0}^{x} dx, \qquad \text{Using} \int \frac{dx}{x^2+a^2} = \frac{1}{a} tan^{-1}({}^{x}/_{a})$$

$$\therefore \quad I = \int_{x=0}^{a} (tan^{-1} 1 - tan^{-1} 0)\,dx = \int_0^a \frac{\pi}{4}\,dx = \frac{\pi}{4}[x]_0^a = \frac{\pi a}{4}$$

3. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$ by changing the order of integration.

Solution:



Given $I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$……..(1)

Here x varies from x = 0 to x = ∞ and y varies from $y = x$ and $y = \infty$.

∴ The region of the integration is the region bounded by the lines x = 0 and y = x.

By changing the order of integration, we get, x varies from $x = 0$ to $x = y$ and y varies from $y = 0$ and $y = \infty$.

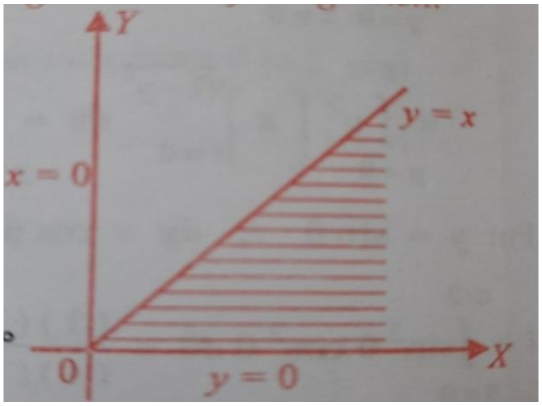
$$\therefore \quad I = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y}\,dx\,dy = \int_{y=0}^{\infty} \frac{e^{-y}}{y}[x]_0^y\,dy$$

$$\therefore \quad I = \int_{y=0}^{\infty} \frac{e^{-y}}{y}.y\,dy = \int_{y=0}^{\infty} e^{-y}\,dy = [-e^{-y}]_0^\infty = -(0-1) = 1$$

4. Evaluate $\int_0^\infty \int_0^x xe^{-x^2/y} dy\,dx$ by changing the order of integration.

Solution:

Given $I = \int_{x=0}^\infty \int_{y=0}^x xe^{-x^2/y} dy\,dx$



Here x varies from x = 0 to x = ∞ and

y varies from $y = 0$ and y = x.

∴ The region of the integration is the region

bounded by the lines y = 0 and y = x.

By changing the order of integration, we get,

x varies from $x = y$ to x = ∞ and y varies

from $y = 0$ and $y = \infty$.

$I = \int_{y=0}^\infty \int_{x=y}^\infty xe^{-x^2/y} dx\,dy$ Put $\frac{x^2}{y} = t$ ∴ $\frac{2x}{y} dx = dt$ $or\ x\,dx = \frac{y}{2}\,dt$

Also when $x = y,\ t = y\ and\ when\ x = \infty,\ t = \infty$

∴ $I = \int_{y=0}^\infty \int_{t=y}^\infty e^{-t}\frac{y}{2} dt\,dy = \int_{y=0}^\infty \frac{y}{2}\left[\frac{e^{-t}}{-1}\right]_{t=y}^\infty dy = \frac{-1}{2}\int_{y=0}^\infty y[0 - e^{-y}dy = \frac{1}{2}\int_{y=0}^\infty ye^{-y}dy$

Applying Bernoulli's rule, we get,

I = $\frac{1}{2}\left[y\left(\frac{e^{-y}}{-1}\right) - (1)(e^{-y})\right]_{y=0}^\infty = \frac{1}{2}[(0-0) - (0-1)] = \frac{1}{2}$

HOME WORK:

1. By changing the order of integration evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx$.

2. By changing the order of integration evaluate $\int_0^1 \int_{e^x}^{e} \frac{dydx}{logy}$.

3. By changing the order of integration evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\,dy\,dx$

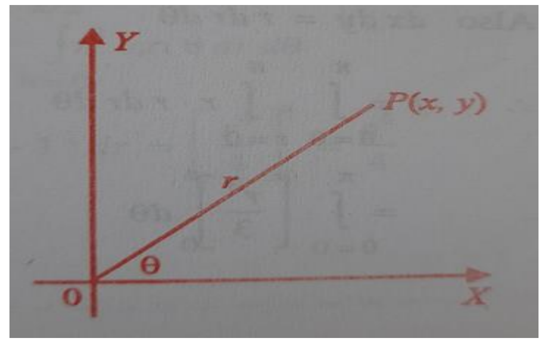
Evaluation of double integral by changing into polar coordinates

To change Cartesian coordinates (x, y) to polar coordinates (r, θ), we have,

$$x = r\cos\theta,\ y = r\sin\theta \quad \therefore \quad r = (x^2 + y^2)^{\frac{1}{2}} \text{ and } dx\,dy = r\,dr\,d\theta.$$

Problems:

1. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing into polar coordinates. Hence show that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.



Solution:

Given $I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$

Here x varies from x = 0 to x = ∞ and y varies from y = 0 to y = ∞.

∴ The region of the integration is the first quadrant in the xy – plane.

In polar coordinates we have $x = r\cos\theta,\ y = r\sin\theta$

$\therefore \quad r = (x^2 + y^2)^{\frac{1}{2}}$ **and** $dx\,dy = r\,dr\,d\theta$.

In the first quadrant θ **varies from** $\theta = 0$ to $\theta = \frac{\pi}{2}$ **and** r varies from $r = 0$ to $r = \infty$.

$\therefore\ I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-r^2} r\,dr\,d\theta$ **Take** $r^2 = t \quad \therefore 2r\,dr = dt$

$\therefore\ r\,dr = \frac{dt}{2}$; When $r = 0, t = 0$ and when $r = \infty,\ t = \infty$.

$\therefore\ I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{t=0}^{\infty} e^{-t} \frac{dt}{2}\,d\theta = \frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{e^{-t}}{-1}\right]_{t=0}^{\infty} d\theta = -\frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}} (0-1)\,d\theta = \frac{1}{2}[\theta]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

$\therefore\ \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \frac{\pi}{4} \quad \therefore \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2}\,dx\,dy = \frac{\pi}{4}$

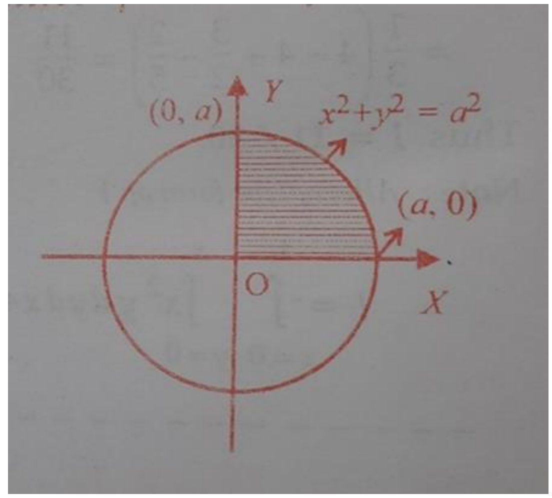
$\therefore\ \int_0^\infty e^{-x^2}\,dx * \int_0^\infty e^{-y^2}\,dy = \frac{\pi}{4}$ **Replace y by x.**

$\therefore\ \int_0^\infty e^{-x^2}\,dx * \int_0^\infty e^{-x}\,dx = \frac{\pi}{4} \quad \therefore \left\{\int_0^\infty e^{-x^2}\,dx\right\}^2 = \frac{\pi}{4}$

Taking square root both sides, we get, $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

2. **Evaluate** $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\ dx\,dy$ **by changing to polar coordinates.**

Solution:



Given I = $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\ dx\,dy$

Here y varies from y = 0 to y = a and x varies from x = 0 to x = $\sqrt{a^2-y^2}$ $i.e.,\ x^2 = a^2 - y^2$

$i.e.,\ x^2+y^2 = a^2$ **is the circle with center at origin and radius a.**

∴ The region of the integration is the first quadrant in the xy – plane.

In polar coordinates we have $x = r\cos\theta,\ y = r\sin\theta$

$\therefore\ r = (x^2+y^2)^{\frac{1}{2}}$ **and** $dx\,dy = r\,dr\,d\theta$.

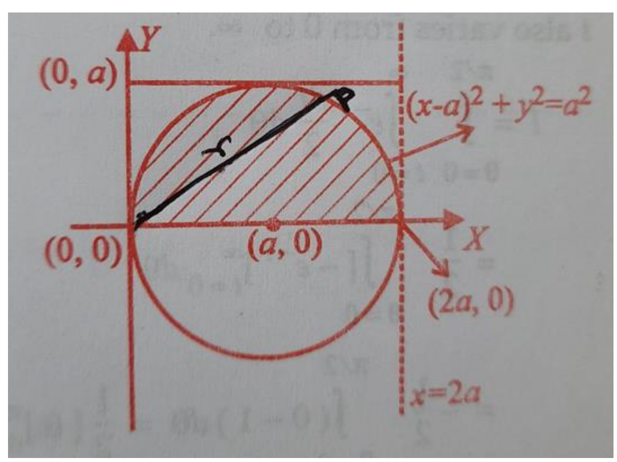
In the first quadrant θ **varies from** $\theta = 0$ *to* $\theta = \frac{\pi}{2}$ **and r** *varies from* $r = 0$ *to* $r = a$.

$$\therefore\ I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} r\sin\theta\,.r.r\ dr\,d\theta = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} r^3 \sin\theta\ dr\,d\theta = \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_{t=0}^{a} \sin\theta\ d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{a^4}{4} - 0\right] \sin\theta\ d\theta = \frac{a^4}{4}\left[-\cos\theta\right]_{t=0}^{\frac{\pi}{2}} = -\frac{a^4}{4}[0-1] = \frac{a^4}{4}$$

3. **Evaluate** $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dx\,dy}{x^2+y^2}$ **by changing into polar coordinates.**

Solution:



Given I = $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dx\,dy}{x^2+y^2}$

Here x varies from x = 0 to x = 2 and y varies from y = 0 to y = $\sqrt{2x-x^2}$.

(i.e., x^2+y^2 **- 2x = 0)**

∴ The region of the integration is the upper half of the circle x^2+y^2 **- 2x = 0 whose center is (1, 0) with radius 1 and passing through the**

origin (0, 0) in the first quadrant of the xy – plane. (Ref. the figure with a = 1)

In polar coordinates we have, $x = r\cos\theta,\ y = r\sin\theta$

$\therefore\ r = (x^2 + y^2)^{\frac{1}{2}}$ and $dx\,dy = rdr\,d\theta.$

Now $y = \sqrt{2x - x^2}$ implies that $y^2 = 2x - x^2\ \therefore\ x^2 + y^2 = 2x$

Put $x = r\cos\theta,\ y = r\sin\theta\ \therefore\ r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$

$\therefore\ r^2 = 2r\cos\theta\ \therefore\ r = 2\cos\theta$

$\therefore$ In the first quadrant θ varies from $\theta = 0$ to $\theta = \frac{\pi}{2}$ and r varies from $r = 0$ to

$r = 2\cos\theta.$

$$\therefore\ \mathrm{I} = \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} \frac{(r\cos\theta)r\,dr\,d\theta}{r^2} = \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}(\cos\theta)dr\,d\theta = \int_0^{\frac{\pi}{2}}(\cos\theta)[r]_0^{2\cos\theta}\,d\theta$$

$$\therefore\ I = \int_0^{\frac{\pi}{2}}\cos\theta\,(2\cos\theta - 0)d\theta = \int_0^{\frac{\pi}{2}} 2\cos^2\theta\,d\theta = \int_0^{\frac{\pi}{2}}(1 + \cos 2\theta)d\theta$$

$$= \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = \left(\frac{\pi}{2} - 0\right) + \frac{1}{2}(0 - 0) = \frac{\pi}{2}$$

HOME WORK:

1. Evaluate $\int_0^a\int_0^{\sqrt{a^2-y^2}}(x^2 + y^2)\ dx\,dy$ by changing to polar coordinates

2. Evaluate $\int_{-a}^a\int_0^{\sqrt{a^2-x^2}}\sqrt{x^2 + y^2}\ dy\,dx$ by changing to polar coordinates

BETA AND GAMMA FUNCTIONS:

Beta Function:

The beta function is defined as $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ …….(1)

where m > 0, n > 0.

Alternative form of Beta function:

(i) Prove that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \; \cos^{2n-1}\theta \;\; d\theta$

Proof:

Put $x = sin^2\theta$ in (1) $\quad\therefore dx = 2\sin\theta\, cos\theta\, d\theta$.

When x = 0, θ = 0; When x = 1, $\theta = \frac{\pi}{2}$.

$$\therefore\ \beta(m,n) = \int_0^{\pi/2}(sin^2\theta)^{m-1}(1-sin^2\theta)^{n-1}\,2\sin\theta\, cos\theta\, d\theta$$

$$\therefore\ \beta(m,n) = \int_0^{\pi/2}(sin^2\theta)^{m-1}(cos^2\theta)^{n-1}\,2\sin\theta\, cos\theta\, d\theta$$

$$\therefore\ \beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta \;\; cos^{2n-1}\theta \;\; d\theta \text{ ………(2)}$$

This is another form of $\beta(m,n)$

Property:

(ii) Prove that $\beta(m,n) = \beta(n,m)$

Proof:

Put $x = 1 - y$ in (1) $\quad\therefore dx = -dy$. When x = 0, y= 1; When x = 1, y= 0.

$$\therefore\ \beta(m,n) = -\int_1^0 (1-y)^{m-1}y^{n-1}\,dy = \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = \beta(n,m)$$

$$\therefore\ \beta(m,n) = \beta(n,m)$$

Gamma Function:

The gamma function is defined as $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx$ ………(3), where n > 0.

Alternative form of Gamma function:

(i) Prove that $\Gamma(n) = 2\int_0^\infty e^{-y^2}y^{2n-1}\,dy$

Proof:

Put $x = y^2$ $in\ (3)$ $\therefore\ dx = 2ydy$. When x = 0, y= 0; When x = ∞, y= ∞.

$\therefore\ \Gamma(n) = \int_0^\infty e^{-y^2}(y^2)^{n-1}\, 2y dy = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$

$\therefore\ \Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$. This is another form of $\Gamma(n)$.

(ii) Prove that $\Gamma(1) = 1$

Proof:

Put n = 1 in (3) $\therefore \Gamma(1) = \int_0^\infty e^{-x}\, dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = -(0-1) = 1 \quad \therefore \Gamma(1) = 1$

Relations:

1. Reduction formula for $\Gamma(n)$: Prove that $\Gamma(n+1) = n\,\Gamma(n)$, n > 0.

Proof:

We have $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$. Replace n by n + 1

$\therefore \Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx$. Using integration by parts we get,

$\Gamma(n+1) = \left[x^n\left(\frac{e^{-x}}{-1}\right)\right]_0^\infty - \int_0^\infty \left(\frac{e^{-x}}{-1}\right) n x^{n-1}\, dx$

$\therefore\ \Gamma(n+1) = (0-0) + n\int_0^\infty e^{-x} x^{n-1}\, dx = n\,\Gamma(n)$

$\therefore \Gamma(n+1) = n\,\Gamma(n)$ ....(4)

2. Prove that $\Gamma(n+1) = n!$, where n is a positive integer.

Proof:

We have $\Gamma(n+1) = n\,\Gamma(n)$ and $\Gamma(1) = 1$

$\therefore \Gamma(2) = \Gamma(1+1) = 1.\ \Gamma(1) = 1.1 = 1!$

$\Gamma(3) = \Gamma(2+1) = 2\,\Gamma(2) = 2.1! = 2!$

$\Gamma(4) = \Gamma(3+1) = 3\,\Gamma(3) = 3.2! = 3!$

....

....

$\therefore\quad \Gamma(n+1) = n!$ ....(5)

Note:

(i) Formula (4) is valid for all the positive values of n.

(ii) Formula (5) is valid for all the positive integer n.

(iii) From (4) we write $\Gamma(n) = \frac{\Gamma(n+1)}{n}$. This formula is valid for all negative non-integer.

(iv) $\Gamma(n)$ is not defined for n = 0 or negative integer

Relation between Beta and Gamma functions:

3. Prove that $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Proof:

We have $\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx$.........(1)

$Similarly,\ \Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$...........(2)

And $\beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta\ \ cos^{2n-1}\theta\ \ d\theta$ ....(3)

Multiplying (1) and (2) we get,

$\Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\, dx\, dy.$

Here x varies from x = 0 to x = ∞ and y varies from y = 0 to y = ∞.

∴ The region of the integration is the first quadrant in the xy – plne

Take $x = r\cos\theta,\ y = r\sin\theta\quad \therefore\ \ r = (x^2+y^2)^{1/2}$ and $dx\, dy = rdr\, d\theta.$

Where θ varies from $\theta = 0\ \ to\ \ \theta = \frac{\pi}{2}$ and r $varies\ from\ \ r = 0\ \ to\ \ r = \infty$.

$\therefore\ \ \Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2} r^{2m-1} cos^{2m-1}\ \theta . r^{2n-1} sin^{2n-1}\ \theta\ rd\theta\, dr.$

$\therefore\ \ \Gamma(m)\Gamma(n) = 2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr * 2\int_0^{\frac{\pi}{2}} cos^{2m-1}\ \theta\ sin^{2n-1}\ \theta\, d\theta$

$\therefore \Gamma(m)\Gamma(n) = \Gamma(m+n)\beta(m,n).\ \ \therefore\ \ \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$......(4), using (1) and (3).

4. Prove that $\Gamma(1/2) = \sqrt{\pi}$ and show that $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$.

Proof:

We have $\beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta\ .cos^{2n-1}\theta\ d\theta$

Put m = n = 1/2

$\therefore\ \ \beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} sin^{2.\frac{1}{2}-1}\theta\ .cos^{2.\frac{1}{2}-1}\theta\ d\theta = 2\int_0^{\pi/2} 1\, d\theta\ = 2\,[\theta]_0^{\pi/2} = 2.\frac{\pi}{2} = \pi$

$\therefore\ \beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi$ ……..(1)

we know that

$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Put m = n = 1/2

$\therefore\ \beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)^2}{\Gamma(1)} = \Gamma\left(\frac{1}{2}\right)^2$ …….(2) $\because\ \Gamma(1) = 1$

∴ From (1) and (2), we get,

$\Gamma\left(\frac{1}{2}\right)^2 = \pi \quad \therefore \quad \Gamma(1/2) = \sqrt{\pi}$

Further we have $\Gamma(n) = 2\int_0^\infty e^{-x^2}x^{2n-1}\,dx$. Put n= $\frac{1}{2}$

$\therefore \sqrt{\pi} = 2\int_0^\infty e^{-x^2}\,dx \quad \therefore \quad \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

Alternative Proof:

We have $\Gamma(n) = 2\int_0^\infty e^{-x^2}x^{2n-1}\,dx$. Put n= $\frac{1}{2}$

$\therefore \Gamma(1/2) = 2\int_0^\infty e^{-x^2}\,dx$……….(1). Replace x by y.

$\therefore \Gamma(1/2) = 2\int_0^\infty e^{-y^2}\,dy$……….. (2). Multiplying (1) and (2) we get,

$[\Gamma(1/2)]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$

Here x varies from x = 0 to x = ∞ and y varies from y = 0 to y = ∞.

∴ The region of the integration is the first quadrant in the xy – plne.

Take $x = r\cos\theta,\ y = r\sin\theta \quad \therefore \quad r = (x^2+y^2)^{1/2}$ and $dx\,dy = r\,dr\,d\theta$.

In the first quadrant θ varies from $\theta = 0$ *to* $\theta = \frac{\pi}{2}$ and r *varies from*

$r = 0$ *to* $r = \infty$.

$\therefore [\Gamma(1/2)]^2 = 4\int_0^{\frac{\pi}{2}}\int_{r=0}^\infty e^{-r^2} r\,dr\,d\theta$. Take $r^2 = t \quad \therefore 2r\,dr = dt \quad \therefore\ r\,dr = \frac{dt}{2}$;

When $r = 0,\ t = 0$ *and when* $r = \infty,\ t = \infty$.

$\therefore [\Gamma(1/2)]^2 = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{t=0}^\infty e^{-t}\frac{dt}{2}\,d\theta = 2\int_{\theta=0}^{\frac{\pi}{2}}[-e^{-t}]_{t=0}^\infty d\theta = -2\int_{\theta=0}^{\frac{\pi}{2}}(0-1)\,d\theta$

$\therefore \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2[\theta]_0^{\frac{\pi}{2}} = 2\left[\frac{\pi}{2} - 0\right] = \pi \quad \therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Note:

We have, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \ \cos^{2n-1}\theta \ \ d\theta$ **……….(1)**

Take 2m – 1 = p and 2n – 1 = q . $\therefore m = \frac{p+1}{2}, \ n = \frac{q+1}{2}$

$\therefore \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \ \cos^q\theta \ \ d\theta$

$\therefore \int_0^{\frac{\pi}{2}} \sin^p x \cos^q x \ dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}, \quad p > -1, \ q > -1$

In particular if q = 0 and p = n, we get $\int_0^{\frac{\pi}{2}} \sin^n x \ dx = \frac{1}{2}\beta\left(\frac{n+1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}$

Similarly, $\int_0^{\frac{\pi}{2}} \cos^n x \ dx = \frac{1}{2}\beta\left(\frac{n+1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}$

Problems:

1. Find the values of $\Gamma(6)$, $\Gamma(-1/2)$, $\Gamma(3/2)$, $\Gamma(-3/2)$, $\Gamma(5/2)$ **and** $\Gamma(-5/2)$.

Solution:

Using $\Gamma(n+1) = n!$ **, We get** $\Gamma(6) = \Gamma(5+1) = 5! = 120$.

Using $\Gamma(n) = \frac{\Gamma(n+1)}{n}$ **, we get** $\Gamma(-1/2) = \frac{\Gamma(-1/2+1)}{-1/2} = -2\,\Gamma(1/2) = -2\sqrt{\pi}$.

Using $\Gamma(n+1) = n\,\Gamma(n)$, **we get,** $\Gamma(3/2) = \Gamma(1/2 + 1) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$.

Using $\Gamma(n) = \frac{\Gamma(n+1)}{n}$, **we get,** $\Gamma(-3/2) = \frac{\Gamma(-3/2+1)}{-3/2} = \frac{\Gamma(-1/2)}{-3/2}$

$\therefore \Gamma(-3/2) = \frac{-2}{3}\frac{\Gamma(-1/2+1)}{-1/2} = \frac{4}{3}\Gamma(1/2) = \frac{4}{3}\sqrt{\pi}$.

Using $\Gamma(n+1) = n\,\Gamma(n)$, **we get,** $\Gamma(5/2) = \Gamma(3/2 + 1) = \frac{3}{2}\Gamma(3/2) = \frac{3}{2}.\Gamma(1/2 + 1)$

$\therefore \Gamma(5/2) = \frac{3}{2}.\frac{1}{2}\Gamma(1/2) = \frac{3}{4}\sqrt{\pi}$.

Using $\Gamma(n) = \frac{\Gamma(n+1)}{n}$, we get, $\Gamma\left(-{}^{5}/_{2}\right) = \frac{\Gamma\left(-{}^{5}/_{2}+1\right)}{-{}^{5}/_{2}} = \frac{\Gamma(-{}^{3}/_{2})}{-{}^{5}/_{2}} = \frac{-2}{5}\frac{\Gamma(-{}^{3}/_{2}+1)}{-{}^{3}/_{2}}$

$\therefore\ \Gamma\left(-{}^{5}/_{2}\right) = \frac{4}{15}\Gamma(-{}^{1}/_{2}) = \frac{4}{15}\frac{\Gamma(-{}^{1}/_{2}+1)}{-{}^{1}/_{2}} = \frac{-8}{15}\Gamma\left({}^{1}/_{2}\right) = \frac{-8}{15}\sqrt{\pi}.$

2. Find the values of $\beta(5,6)$ and $\beta(-{}^{1}/_{2}, {}^{3}/_{2})$.

Solution:

Using $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, we get, $\beta(5,6) = \frac{\Gamma(5)\Gamma(6)}{\Gamma(11)} = \frac{(4!)(5!)}{10!} = \frac{(24)(5!)}{10{,}9{,}8{,}7{,}6{,}(5!)} = \frac{1}{1260}$

And $\beta(-{}^{1}/_{2}, {}^{3}/_{2}) = \frac{\Gamma(-{}^{1}/_{2})\Gamma({}^{3}/_{2})}{\Gamma(-{}^{1}/_{2}+{}^{3}/_{2})} = \frac{\frac{\Gamma\left(-\frac{1}{2}+1\right)}{\left(-\frac{1}{2}\right)}\Gamma\left(\frac{1}{2}+1\right)}{\Gamma(1)} = -2\Gamma\left(\frac{1}{2}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = -\sqrt{\pi}.\sqrt{\pi} = -\pi.$

3. *Prove that* $\int_0^{\frac{\pi}{2}}\sqrt{\tan\theta}\,d\theta = \frac{1}{2}\Gamma({}^{1}/_{4})\Gamma({}^{3}/_{4})$.

Solution:

$\int_0^{\frac{\pi}{2}}\sqrt{\tan\theta}\,d\theta = \int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin\theta}{\cos\theta}}\,d\theta = \int_0^{\frac{\pi}{2}}(\sin\theta)^{\frac{1}{2}}(\cos\theta)^{-\frac{1}{2}}\,d\theta$

$= \frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right)$, *using* $\int_0^{\frac{\pi}{2}}\sin^m x\cos^n x\,dx = \frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$

$= \frac{1}{2}\frac{\Gamma({}^{3}/_{4})\Gamma({}^{1}/_{4})}{\Gamma({}^{3}/_{4}+{}^{1}/_{4})} = \frac{1}{2}\frac{\Gamma({}^{3}/_{4})\Gamma({}^{1}/_{4})}{\Gamma(1)}$, using $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$= \frac{1}{2}\Gamma({}^{1}/_{4})\Gamma({}^{3}/_{4})$. Using $\Gamma(1) = 1$

4. *Prove that* $\int_0^{\frac{\pi}{2}}\sqrt{\sin\theta}\,d\theta * \int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{\sin\theta}} = \pi$.

Solution:

Let $I = \int_0^{\frac{\pi}{2}}\sqrt{\sin\theta}\,d\theta * \int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2}\sin^{\frac{1}{2}}\theta\,d\theta * \int_0^{\frac{\pi}{2}}\sin^{-\frac{1}{2}}\theta\,d\theta$

$= \frac{1}{2}\beta\left[\frac{\frac{1}{2}+1}{2}, \frac{1}{2}\right] * \frac{1}{2}\beta\left[\frac{-\frac{1}{2}+1}{2}, \frac{1}{2}\right] = \frac{1}{4}\beta\left(\frac{3}{4},\frac{1}{2}\right) * \beta\left[\frac{1}{4}, \frac{1}{2}\right] = \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)} * \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$

$= \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{1}{4}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}$ $\because\ \Gamma\left(\frac{5}{4}\right) = \Gamma\left(\frac{1}{4}+1\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$

$\therefore\ I = \Gamma\left(\frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\sqrt{\pi} = \pi.$

5. Prove that $\int_0^1 x^3\left(1-\sqrt{x}\right)^5 dx = 2\beta(8,6)$.

Solution:

Take $\sqrt{x} = sin^2\theta$ or $x = sin^4\theta$ $\therefore$ dx= $4\, sin^3\theta cos\theta\, d\theta$.

When x = 0, $\theta = 0$; When x = 1, $\theta = \frac{\pi}{2}$.

$$\therefore \int_0^1 x^3\left(1-\sqrt{x}\right)^5 dx = \int_0^{\frac{\pi}{2}} sin^{12}\theta(1-sin^2\theta)^5 4\, sin^3\theta cos\theta\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} sin^{15}\theta\, cos^{10}\theta\, cos\theta\, d\theta = 4\int_0^{\frac{\pi}{2}} sin^{15}\theta\, cos^{11}\theta d\theta$$

$$\therefore \int_0^1 x^3\left(1-\sqrt{x}\right)^5 dx = 4.\frac{1}{2}\beta\left(\frac{15+1}{2},\frac{11+1}{2}\right) = 2\beta(8,6)$$

6. *Prove that* $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{4}\frac{\Gamma(1/4)}{\Gamma(3/4)}$.

Solution:

Take $x^4 = sin^2\theta$ or $x = sin^{\frac{1}{2}}\theta$ $\therefore$ dx= $\frac{1}{2}sin^{\frac{-1}{2}}\theta cos\theta\, d\theta$.

When x = 0, $\theta = 0$; When x = 1, $\theta = \frac{\pi}{2}$.

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{sin^{\frac{-1}{2}}\theta cos\theta\, d\theta}{\sqrt{1-sin^2\theta}} = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin^{\frac{-1}{2}}\theta\, d\theta$$

$$= \frac{1}{2}.\frac{1}{2}\beta\left(\frac{\frac{-1}{2}+1}{2},\frac{1}{2}\right) = \frac{1}{4}\beta\left(\frac{1}{4},\frac{1}{2}\right), \text{ using } \int_0^{\frac{\pi}{2}} sin^n x\, dx = \frac{1}{2}\beta\left(\frac{n+1}{2},\frac{1}{2}\right)$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} = \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4}\frac{\Gamma(1/4)}{\Gamma(3/4)}, \text{ using } \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

7. Prove that $\int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\sqrt{\pi}\ \Gamma(1/4)}{4\sqrt{2}\ \Gamma(3/4)}$.

Solution:

$$I = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

Take $x^4 = tan^2\theta$ or $x = tan^{1/2}\theta$ $\therefore$ dx= $\frac{1}{2}tan^{-\frac{1}{2}}\theta\, sec^2\theta d\theta$

When x=0, $tan^2\theta = 0$ $\therefore \theta = 0$; *When* $x = 1$, $tan^2\theta = 1$ $\therefore \theta = \frac{\pi}{4}$.

$$\therefore I = \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{tan^{-\frac{1}{2}}\theta\, sec^2\theta d\theta}{\sqrt{1+tan^2\theta}} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{tan^{-\frac{1}{2}}\theta\, sec^2\theta d\theta}{sec\theta}$$

$\therefore$ $I = \frac{1}{2}\int_0^{\frac{\pi}{4}} tan^{-\frac{1}{2}}\theta\ sec\theta\ d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{sin^{-\frac{1}{2}}\theta\ d\theta}{cos^{-\frac{1}{2}}\theta cos\theta} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{sin^{-\frac{1}{2}}\theta\ d\theta}{cos^{\frac{1}{2}}\theta} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{sin^{\frac{1}{2}}\theta cos^{\frac{1}{2}}\theta}$

$\therefore$ $I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{sin\theta cos\theta}} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\frac{sin2\theta}{2}}} = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{sin2\theta}}$

Now take $2\theta = \emptyset$ $\therefore \theta = \frac{\emptyset}{2}$ $\therefore d\theta = \frac{d\emptyset}{2}$.

When $\theta = 0$, $\emptyset = 0$; $when\ \theta = \frac{\pi}{4}$, $\emptyset = \frac{\pi}{2}$.

$\therefore$ $I = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}d\emptyset}{\sqrt{sin\emptyset}} = \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} sin^{-\frac{1}{2}}\emptyset\ cos^0\emptyset d\emptyset$. Using $\int_0^{\frac{\pi}{2}} sin^p x\ cos^q x\ dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$,

$I = \frac{1}{2\sqrt{2}}.\frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4\sqrt{2}}\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(1/4+1/2)} = \frac{1}{4\sqrt{2}}\frac{\Gamma(1/4)\sqrt{\pi}}{\Gamma(3/4)}$.

8. *Prove that* $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{4}\frac{\Gamma(3/4)}{\Gamma(5/4)}$.

Solution:

Let $I = \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} dx$

Take $x^4 = sin^2\theta\ or\ x = sin^{\frac{1}{2}}\theta$ $\therefore dx = \frac{1}{2}sin^{-\frac{1}{2}}\theta\ cos\theta d\theta$

When $x = 0$, $\theta = 0$; When $x = 1$, $\theta = \frac{\pi}{2}$.

$\therefore$ $I = \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{sin\theta\ sin^{\frac{-1}{2}}\theta cos\theta\ d\theta}{\sqrt{1-sin^2\theta}} = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin^{\frac{1}{2}}\theta\ cos^0\theta\ d\theta$

$\therefore$ $I = \frac{1}{2}.\frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2}, \frac{1}{2}\right) = \frac{1}{4}\beta\left(\frac{3}{4}, \frac{1}{2}\right)$, Using $\int_0^{\frac{\pi}{2}} sin^p x\ cos^q x\ dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$I = \frac{1}{4}\frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(3/4+1/2)} = \frac{1}{4}\frac{\Gamma(3/4)\sqrt{\pi}}{\Gamma(5/4)}$

9. *Prove that* $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} * \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$

Solution:

Let $I_1 = \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} dx$

Take $x^4 = sin^2\theta\ or\ x = sin^{\frac{1}{2}}\theta$ $\therefore dx = \frac{1}{2}sin^{-\frac{1}{2}}\theta\ cos\theta d\theta$

When $x = 0,\ \theta = 0$; When $x = 1,\ \theta = \frac{\pi}{2}$.

$\therefore\ I_1 = \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin\theta\,\sin^{\frac{-1}{2}}\theta\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\,d\theta$

$\therefore\ I_1 = \frac{1}{2}.\frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2},\ \frac{1}{2}\right) = \frac{1}{4}\beta\left(\frac{3}{4},\frac{1}{2}\right)$……(1). Using $\int_0^{\frac{\pi}{2}} \sin^n x\ dx = \frac{1}{2}\beta\left(\frac{n+1}{2},\frac{1}{2}\right)$

Let $I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

Take $x^4 = \tan^2\theta$ or $x = \tan^{1/2}\theta$ $\therefore$ dx= $\frac{1}{2}\tan^{-\frac{1}{2}}\theta\,\sec^2\theta d\theta$

When x=0, $\tan^2\theta = 0\ \therefore \theta = 0$; When $x = 1$, $\tan^2\theta = 1\ \therefore \theta = \frac{\pi}{4}$.

$\therefore\ I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\tan^{-\frac{1}{2}}\theta\,\sec^2\theta d\theta}{\sqrt{1+\tan^2\theta}} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\tan^{-\frac{1}{2}}\theta\,\sec^2\theta d\theta}{\sec\theta}$

$\therefore\ I_2 = \frac{1}{2}\int_0^{\frac{\pi}{4}} \tan^{-\frac{1}{2}}\theta\,\sec\theta\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\sin^{-\frac{1}{2}}\theta\,d\theta}{\cos^{-\frac{1}{2}}\theta\cos\theta} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\sin^{-\frac{1}{2}}\theta\,d\theta}{\cos^{\frac{1}{2}}\theta} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sin^{\frac{1}{2}}\theta\cos^{\frac{1}{2}}\theta}$

$\therefore\ I_2 = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\sin\theta\cos\theta}} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\frac{\sin 2\theta}{2}}} = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\sin 2\theta}}$

Now take $2\theta = \emptyset\ \therefore \theta = \frac{\emptyset}{2}\ \therefore d\theta = \frac{d\emptyset}{2}$.

When $\theta = 0$, $\emptyset = 0$; when $\theta = \frac{\pi}{4}$, $\emptyset = \frac{\pi}{2}$.

$\therefore\ I_2 = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}d\emptyset}{\sqrt{\sin\emptyset}} = \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\emptyset\,d\emptyset$. Using $\int_0^{\frac{\pi}{2}} \sin^n x\ dx = \frac{1}{2}\beta\left(\frac{n+1}{2},\frac{1}{2}\right)$, we get,

$I_2 = \frac{1}{2\sqrt{2}}.\frac{1}{2}\beta\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right)$ ……..(2).

$\therefore \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} * \int_0^1 \frac{dx}{\sqrt{1+x^4}} = I_1 . I_2 = \frac{1}{4}\beta\left(\frac{3}{4},\frac{1}{2}\right).\frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right)$

$\therefore \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} * \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{16\sqrt{2}}\frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(3/4+1/2)}.\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(1/4+1/2)}$, using $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$= \frac{1}{16\sqrt{2}}\frac{\Gamma(3/4)\sqrt{\pi}}{\Gamma(5/4)}.\frac{\Gamma(1/4)\sqrt{\pi}}{\Gamma(3/4)} = \frac{\pi}{16\sqrt{2}}\frac{\Gamma(1/4)}{\Gamma(5/4)}$$

$$= \frac{\pi}{16\sqrt{2}}\frac{\Gamma(1/4)}{\Gamma(1/4+1)} = \frac{\pi}{16\sqrt{2}}\frac{\Gamma(1/4)}{\frac{1}{4}\Gamma(1/4)} = \frac{\pi}{4\sqrt{2}}.$$

10. ***Show that*** $\Gamma(n) = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy$

Solution:

We have $\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx$. Take $x = \log\frac{1}{y}$ $\therefore$ $e^x = \frac{1}{y}$ $\therefore$ $e^{-x} = y$

$\therefore -x = \log y$ $\therefore$ $x = -\log y$ $\therefore$ $dx = -\frac{1}{y}dy$. When $x = 0$, $y = 1$; When $x = \infty$, $y = 0$.

$\therefore \Gamma(n) = -\int_1^0 \left(\log\frac{1}{y}\right)^{n-1} y.\ \frac{1}{y}dy = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy$

11. ***Show that*** $\beta(p,q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy$

Solution:

We have $\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx$

Take $x = \frac{1}{1+y}$ $\therefore$ $dx = \frac{-1}{(1+y)^2}dy$ and $1 - x = 1 - \frac{1}{1+y} = \frac{y}{1+y}$

When $x = 0$, $1 + y = \frac{1}{x} = \infty$ $\therefore$ $y = \infty$; When $x = 1$, $1 + y = 1$ $\therefore$ $y = 0$.

$\therefore \beta(p,q) = -\int_\infty^0 \left(\frac{1}{1+y}\right)^{p-1}\left(\frac{y}{1+y}\right)^{q-1}\frac{1}{(1+y)^2}dy = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dx$

12. ***Show that*** $\beta(p,q) = \int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}} dx$

Solution:

We have $\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx$

Take $x = \frac{1}{1+y}$ $\therefore$ $dx = \frac{-1}{(1+y)^2}dy$ and $1 - x = 1 - \frac{1}{1+y} = \frac{y}{1+y}$.

When $x = 0$, $1 + y = \frac{1}{x} = \infty$ $\therefore$ $y = \infty$; When $x = 1$, $1 + y = 1$ $\therefore$ $y = 0$.

$\therefore \beta(p,q) = -\int_\infty^0 \left(\frac{1}{1+y}\right)^{p-1}\left(\frac{y}{1+y}\right)^{q-1}\frac{1}{(1+y)^2}dy = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dx$

$\therefore \beta(p,q) = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}dy + \int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dy$ ....(1)

Take $y = \frac{1}{t}$ in second integrand on RHS $\therefore$ $dy = \frac{-1}{t^2}dt$.

When $y = 1$, $t = 1$; When $y = \infty$, $t = 0$.

$\therefore \ \beta(p,q) = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}} dy + \int_1^0 \frac{(\frac{1}{t})^{q-1}}{\left(1+\frac{1}{t}\right)^{p+q}} (\frac{-1}{t^2})\text{dt}$

$= \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}} dy + \int_0^1 \frac{1}{\left(\frac{t+1}{t}\right)^{p+q}} (\frac{1}{t^{q-1+2}})\text{dt}$

$\therefore \beta(p,q) = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}} dy + \int_0^1 \frac{t^{p+q}}{(1+t)^{p+q}} (\frac{1}{t^{q+1}})\text{dt}$

$= \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}} dy + \int_0^1 \frac{t^{p-1}}{(1+t)^{p+q}}$ dt. Replace y and t by x

$\therefore \beta(p,q) = \int_0^1 \frac{x^{q-1}}{(1+x)^{p+q}} dy + \int_0^1 \frac{x^{p-1}}{(1+x)^{p+q}}\text{dt} = \int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}} dx.$

13. Evaluate $\int_0^\infty \frac{x^c}{c^x} dx$

Solution:

Take c^x=e^y $\therefore$ $log c^x$=$log e^y$ $\therefore$ x logc = y loge $\therefore$ x logc = y $\therefore$ x = $\frac{1}{logc}$ y

Differentiating we get, dx = $\frac{1}{logc}$dy. When x = 0, y = 0; x = ∞, y = ∞.

$\therefore \int_0^\infty \frac{x^c}{c^x} dx = \int_0^\infty \left(\frac{1}{logc} y\right)^c \frac{1}{e^y} \frac{1}{logc} \text{dy} = \frac{1}{(logc)^c} \frac{1}{logc} \int_0^\infty y^c e^{-y} dy = \frac{1}{(logc)^{c+1}} \int_0^\infty e^{-y} y^c dy$

$= \frac{1}{(logc)^{c+1}} \Gamma(c+1) = \frac{\Gamma(c+1)}{(logc)^{c+1}}$ Using $\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$

14. Evaluate $\int_0^\infty a^{-bx^2} dx$

Solution:

Take a^{-bx^2}=e^{-t} Take log on both sides. $\therefore$ (−b log a) $x^2 = -t$ $\therefore$ (b log a) $x^2 = t$

$\therefore$ (b log a) $2x.dx = dt$ $\therefore$ $dx = \frac{1}{2(b\log a)x} dt = \frac{1}{2(b\log a)\sqrt{\frac{t}{b.loga}}} dt$

$\therefore \int_0^\infty a^{-bx^2} dx = \int_0^\infty e^{-t} \frac{1}{2(bloga)\sqrt{\frac{t}{b.loga}}} dt = \frac{1}{2\sqrt{(bloga)}} \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$

$= \frac{1}{2\sqrt{(bloga)}} \Gamma\left(-\frac{1}{2}+1\right)$ Using $\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$

$\therefore \int_0^\infty a^{-bx^2} dx = \frac{\Gamma\left(\frac{1}{2}\right)}{2\sqrt{bloga}} = \frac{\sqrt{\pi}}{2\sqrt{(bloga)}}.$

15. Evaluate $\int_0^1 x^5 \left(\log\frac{1}{x}\right)^3 dx$

Solution:

Take $\log\frac{1}{x} = t$ $\therefore$ $x = e^{-t}$ $\therefore$ $dx = -e^{-t}dt$

When $x=0$, $t = \infty$; When $x = 1$, $t = 0$.

$$\therefore \int_0^1 x^5 \left(\log\frac{1}{x}\right)^3 dx = \int_\infty^0 e^{-5t}\, t^3 (-e^{-t}) dt = -\int_\infty^0 e^{-6t}\, t^3 dt = \int_0^\infty e^{-6t}\, t^3 dt$$

Put $6t = y$ $\therefore$ $6\,dt = dy$ $\therefore$ $dt = \frac{1}{6}dy$

$$\therefore \int_0^1 x^5 \left(\log\frac{1}{x}\right)^3 dx = \int_0^\infty e^{-y}\left(\frac{y}{6}\right)^3 \frac{dy}{6} = \frac{1}{6^4}\int_0^\infty e^{-y} y^3\, dy = \frac{1}{6^4}\Gamma(4) = \frac{1}{1296}(3!) = \frac{1}{216}$$

HOME WORK:

1. Evaluate (i) $\Gamma(7/2)$ (ii) $\beta\left(\frac{1}{4},\frac{1}{2}\right)$ (iii) $\beta\left(\frac{7}{2},-\frac{1}{2}\right)$
2. Express the following integrals in terms of gamma functions

   (i) $\int_0^\infty e^{-x^2}dx$ (ii) $\int_0^\infty x^{p-1}e^{-kx}dx$, $(k > 0)$
3. Show $\int_0^\infty \frac{x^4}{4^x}dx = \frac{\Gamma(5)}{(\log 4)^5}$.
4. Given $\int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}$, show that $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

   hence evaluate $\int_0^\infty \frac{dy}{1+y^4}$.
5. Prove that $\int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right)$.
6. Prove that $\int_0^1 x^m(\log x)^n dx = \frac{(-1)^n\, n!}{(m+1)^{n+1}}$, where n is a positive integer and m>-1.

   Hence evaluate $\int_0^1 x(\log x)^3 dx$.