ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER

Linear and Bernoulli's differential equations. Exact and reducible to exact differential equations. Applications of ODE's-Orthogonal trajectories, Newton's law of cooling. Nonlinear differential equations: Introduction to general and singular solutions; Solvable for p only; Clairaut's equations, reducible to Clairaut's equations. Problems.

Solutions of ordinary differential equations of first order and first degree:-

Definition of Differential Equation:

An equation involving a function of one or more independent variable and its derivatives is called a differential equation.

Order of the Differential Equation:

The highest derivative present in the differential equation is called order of the differential equation.

Degree of the Differential Equation:

The highest power of the highest derivative present in the differential equation is called degree of the differential equation.

Linear Differential Equation of First Order and First Degree:

Definition:

A differential equation of the form $\frac{dy}{dx} + \mathbf{P\,y = Q}$, where **P** and **Q** are functions of **x** or constants, is called a linear differential equation.

The solution of the linear differential equation is given by $\mathbf{y\,(I.F) = \int Q\,(I.F)\,dx + c}$,
Where $\mathbf{I.F = e^{\int P\,dx}}$ is called the Integrating Factor.

Another form of the linear differential equation is given by $\frac{dx}{dy} + Px = Q$, where **P** and **Q** are functions of **y** or constants and its solution is given by $\mathbf{x\,(I.F) = \int Q\,(I.F)\,dy + c}$, where

$\mathbf{I.F} = e^{\int p\,dy}$.

Problems:

1. Solve $\frac{dy}{dx} - \frac{2y}{x} = x + x^2$.

Solution:

On comparing with $\frac{dy}{dx} + Py = Q$, we get $P = -\frac{2}{x}$, $Q = x + x^2$, which are functions of x only.

$\therefore \quad I.F = e^{\int Pdx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$.

$\therefore$ The solution is $y.(I.F) = \int Q\,(I.F)dx + c$.

$\therefore\ y.\frac{1}{x^2} = \int (x + x^2)\frac{1}{x^2}\ dx + c.$

$\therefore\ \frac{y}{x^2} = \int \left(\frac{1}{x} + 1\right) dx + c.$

$\therefore\ \frac{y}{x^2} = \log x + x + c$ is required solution.

2. Solve $\frac{dy}{dx} + 3x^2y = x^5e^{x^3}$.

Solution:

On comparing with $\frac{dy}{dx} + Py = Q$, we get $P = 3x^2,\ Q = x^5e^{x^3}$, which are functions of x only.

$\therefore\ I.F = e^{\int P\,dx} = e^{\int 3x^2 dx} = e^{x^3}$.

$\therefore$ The solution is $y.(I.F) = \int Q\,(I.F)dx + c.$

$\therefore\ y.e^{x^3} = \int x^5e^{2x^3}\ dx + c. \quad \therefore\ y.e^{x^3} = \int x^3x^2e^{2x^3}\ dx + c.$

Take $x^3 = t. \quad \therefore\ 3x^2dx = dt. \quad \therefore\ x^2dx = \frac{1}{3}dt.$

$\therefore\ y.e^{x^3} = \frac{1}{3}\int t.e^{2t}\ dt + c. \quad \therefore\ y.e^{x^3} = \frac{1}{3}\left[t\frac{e^{2t}}{2} - \frac{e^{2t}}{4}\right] + c$

$\therefore\ y.e^{x^3} = x^3\frac{e^{2x^3}}{6} + \frac{e^{2x^3}}{12} + c$ is the required solution.

3. Solve $(x + 1)\frac{dy}{dx} - y = e^{3x}(x + 1)^2$

Solution:

Dividing the equation both sides by (x+1), we get $\frac{dy}{dx} - \frac{y}{(x+1)} = e^{3x}(x + 1)$

On comparing with $\frac{dy}{dx} + Py = Q$, we get $P = -\frac{1}{(x+1)}$, $Q = e^{3x}(x + 1)$, which are functions of x only.

$\therefore\ I.F = e^{\int P\,dx} = e^{\int -\frac{1}{(x+1)}dx} = e^{-\log(x+1)} = e^{\log(x+1)^{-1}} = e^{\log\left[1/(x+1)\right]} = \frac{1}{x+1}.$

$\therefore$ The solution is $y.(I.F) = \int Q\,(I.F)dx + c.$

$\therefore\ y.\frac{1}{(x+1)} = \int e^{3x}(x + 1).\frac{1}{(x+1)}dx + c. \quad \therefore\ y.\frac{1}{(x+1)} = \int e^{3x}\ dx + c.$

$\therefore\ \frac{y}{(x+1)} = \frac{e^{3x}}{3} + c. \quad \therefore\ y = (x + 1)\left[\frac{e^{3x}}{3} + c\right]$ is the required solution.

4. Solve $x(1-x^2)\frac{dy}{dx}+(2x^2-1)y=x^3$

Solution:

Dividing the equation both sides by $x(1-x^2)$, we get,

$$\frac{dy}{dx}+\frac{2x^2-1}{x(1-x^2)}y=\frac{x^2}{(1-x^2)}$$

On comparing with $\frac{dy}{dx}+Py=Q$, we get $P=\frac{2x^2-1}{x(1-x^2)}$, $Q=\frac{x^2}{(1-x^2)}$, which are functions of x only.

Resolving P into partial fraction, we get, $\frac{2x^2-1}{x(1-x^2)}=\frac{2x^2-1}{x(1-x)(1+x)}=\frac{A}{x}+\frac{B}{1-x}+\frac{C}{1+x}$.

$2x^2-1=A(1-x)(1+x)+Bx(1+x)+Cx(1-x)$.

Put $x=0$. $\therefore\ -1=A$. $\therefore\ A=-1$. Put $x=1$. $\therefore\ 1=2B$. $\therefore\ B=\frac{1}{2}$.

Put $x=-1$. $\therefore\ 1=-2C$. $\therefore\ C=-\frac{1}{2}$.

$$\therefore\ \int p\,dx=-\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{(1-x)}dx-\frac{1}{2}\int\frac{1}{(1+x)}dx=-\log x-\frac{1}{2}\log(1-x)-\frac{1}{2}\log(1+x).$$

$$\therefore\ \int Pdx=-[\log x+\log\sqrt{1-x}+\log\sqrt{1+x}]=-\log\left(x\sqrt{1-x^2}\right)=\log\left(x\sqrt{1-x^2}\right)^{-1}$$

$$\therefore\ I.F=e^{\int P\,dx}=e^{\log\left(x\sqrt{1-x^2}\right)^{-1}}=\frac{1}{x\sqrt{1-x^2}}.$$

$\therefore$ The solution is $y.(I.F)=\int Q\,(I.F)dx+c$.

$$\therefore\ y.\frac{1}{x\sqrt{1-x^2}}=\int\frac{x^2}{(1-x^2)}.\frac{1}{x\sqrt{1-x^2}}\,dx+c.$$

$\therefore\ y.\frac{1}{x\sqrt{1-x^2}}=\int\frac{x}{(1-x^2)^{3/2}}\,dx+c$, Let $(1-x^2)=t$. $\therefore\ 2x\,dx=dt$. $\therefore\ x\,dx=-\frac{dt}{2}$.

$$\therefore\ y.\frac{1}{x\sqrt{1-x^2}}=-\frac{1}{2}\int\frac{dt}{t^{3/2}}+c=-\frac{1}{2}\frac{t^{-1/2}}{(-1/2)}+c.\qquad\therefore\ y.\frac{1}{x\sqrt{1-x^2}}=\frac{1}{t^{1/2}}+c.$$

$\therefore\ y.\frac{1}{x\sqrt{1-x^2}}=\frac{1}{\sqrt{1-x^2}}+c.$ $\therefore\ y=x+c\,x\sqrt{1-x^2}$ is the required solution.

5. Solve $\frac{dy}{dx}+y\cot x=4x\,\mathrm{cosec}\,x$ if $y=0$ when $x=\frac{\pi}{2}$.

Solution:

On comparing with $\frac{dy}{dx}+Py=Q$, we get $P=\cot x$, $Q=4x\,\mathrm{cosec}\,x$, which are functions of x only. $\therefore$ $I.F=e^{\int P\,dx}=e^{\int\cot x\,dx}=e^{\log\sin x}=\sin x$.

$\therefore$ The solution is $y.(I.F) = \int Q\,(I.F)dx + c.$

$\therefore$ $y.\sin x = \int 4\,x\ \text{cosec}\,x\,.\sin x\ dx + c = \int 4\,x\ dx + c.$

$\therefore$ $y.\sin x = 2x^2 + c\ \ \ldots(1)$ is the general solution. Given $y = 0$ when $x = \frac{\pi}{2}$.

$\therefore$ $0 = 2.\left(\frac{\pi^2}{4}\right) + c.\quad \therefore\quad c = -\frac{\pi^2}{2}.$

$\therefore$ $y\sin x = 2x^2 - \frac{\pi^2}{2}$ is the particular solution.

6. Solve $(1 + y^2)\,dx + (x - \tan^{-1}y)dy = 0.$

Solution:

The given equation can be rewritten as $\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$. $\therefore$ $\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$.

This equation is a linear in x of the form $\frac{dx}{dy} + Px = Q.$

$\therefore$ $P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1}y}{1+y^2}$, which are functions of y only.

$\therefore$ $I.F = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}.$

$\therefore$ The solution is $x.(I.F) = \int Q\,(I.F)dy + c.$

$\therefore$ $x\,.\,e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2}.\,e^{\tan^{-1}y}\,dy + c.$ Take $\tan^{-1}y = t.$ $\therefore$ $\frac{1}{1+y^2}dy = dt.$

$\therefore$ $x\,.\,e^{\tan^{-1}y} = \int t.\,e^t\,dt + c.$ $\therefore$ $x\,.\,e^{\tan^{-1}y} = (t.\,e^t - e^t) + c.$

$\therefore$ $x.\,e^{\tan^{-1}y} = (\tan^{-1}y - 1)e^{\tan^{-1}y} + c$ is the required solution.

7. Solve $y\,dx + (3x - xy - 2)dy = 0.$

Solution:

The given equation can be rewritten as $y\,dx = -(3x - xy - 2)dy.$ $\therefore$ $\frac{dx}{dy} + x\left(\frac{3-y}{y}\right) = \frac{2}{y}$.

On comparing with $\frac{dx}{dy} + Px = Q,$ we get $P = \left(\frac{3-y}{y}\right),\ Q = \frac{2}{y}$, which are functions of y only.

$\therefore$ $I.F = e^{\int p\,dy} = e^{\int\left(\frac{3-y}{y}\right)dy} = e^{\int\left(\frac{3}{y}-1\right)dy} = e^{3\log y - y} = e^{\log y^3}.\,e^{-y} = y^3e^{-y}.$

$\therefore$ The solution is $x.(I.F) = \int Q\,(I.F)dy + c.$

$\therefore \quad x\,.\,y^3e^{-y} = \int \frac{2}{y}\,(y^3e^{-y})\,dy + c. \qquad \therefore \quad x\,.\,y^3e^{-y} = 2\int y^2e^{-y}\,dy + c.$

$\therefore \quad x\,.\,y^3e^{-y} = 2\{y^2(-e^{-y}) - (2y)e^{-y} + (2)(-e^{-y})\} + c.$

$\therefore \quad x\,.\,y^3e^{-y} = -2e^{-y}[y^2 + 2y + 2] + c. \qquad \therefore \quad x\,.\,y^3 = -2[y^2 + 2y + 2] + \frac{c}{e^{-y}}\,.$

$\therefore \quad x\,.\,y^3 + 2[y^2 + 2y + 2] = ce^y$ is the required solution.

8. Solve $dr + (2r\cot\theta + \sin 2\theta)d\theta = 0.$

Solution:

The given equation can be rewritten as $\frac{dr}{d\theta} + (2\cot\theta)r = -\sin 2\theta.$

Here $P = 2\cot\theta\,,\ \ Q = -\sin 2\theta = -2\sin\theta\cos\theta.$

$\therefore \quad I.F = e^{\int P d\theta} = e^{\int 2\cot\theta\, d\theta} = e^{\log \sin^2\theta} = \sin^2\theta.$

$\therefore$ The solution is $r.(I.F) = \int Q\,(I.F)\,d\theta + c.$

$\therefore \quad r\sin^2\theta = \int -2\sin\theta\cos\theta\,(\sin^2\theta)d\theta + c.$

$\therefore \quad r\sin^2\theta = -2\int \sin^3\theta\cos\theta\, d\theta + c.$ Let $\sin\theta = t.$ $\therefore \cos\theta\, d\theta = dt.$

$\therefore \quad r\sin^2\theta = -2\int t^3\,dt + c. \qquad \therefore \quad r\sin^2\theta = -2\frac{t^4}{4} + c.$

$\therefore \quad r\sin^2\theta = -\frac{(\sin\theta)^4}{2} + c. \qquad \therefore \quad 2r\sin^2\theta + \sin^4\theta = 2c$ is the required solution.

HOME WORK:

1. **Solve $x\frac{dy}{dx} + y = \log x.$**

2. **Solve $(1 - x^2)\frac{dy}{dx} + xy = ax.$**

3. **Solve $x\cos x\frac{dy}{dx} + (x\sin x + \cos x)y = 1.$**

4. **Solve $(x + 2y^3)\frac{dy}{dx} = y.$**

5. **Solve the differential equation: $\frac{dy}{dx} + \frac{2x}{(1+x^2)}y = \frac{1}{(1+x^2)^2}$ given $y = 0$, when $x = 1$.**

Bernoulli's Equation:

An equation of the form $\frac{dy}{dx} + Py = Qy^n$, where **P** & **Q** are functions of **x** or constants, is called as Bernoulli's Equation.

This equation can be reducible to the linear differential equation of the form $\frac{\mathbf{dy}}{\mathbf{dx}} + \mathbf{Py} = \mathbf{Q}$ and its solution is obtained as follows.

Method of Finding Solution to Bernoulli's Equation:

1. Arrange the given equation in the form $\frac{dy}{dx} + Py = Qy^n$
2. Divide the equation throughout by y^n to obtain $\frac{1}{y^n}\frac{dy}{dx} + P\frac{1}{y^{n-1}} = Q$……. (1)
3. Take substitution $\frac{1}{y^{n-1}} = t$ and differentiate w.r.t. x.
4. Substitute in the above equation (1) and reduce it in the form $\frac{dt}{dx} + Pt = Q$, which is a linear equation in t.
5. Find Integrating factor $\mathrm{I.F} = e^{\int P\,dx}$
6. The solution is $t\,(\mathrm{I.F}) = \int Q\,(\mathrm{I.F})\,dx + c$

Note: Another form of the Bernoulli's equation is $\frac{dx}{dy} + Px = Qx^n$, where P and Q are functions of y.

Problems:

1. Solve $x\frac{dy}{dx} + y = x^3y^6$.

Solution:

Given $x\frac{dy}{dx} + y = x^3y^6$.

Dividing both the sides by x, we get, $\frac{dy}{dx} + \frac{1}{x}y = x^2y^6$ ………. (1).

This is of the form $\frac{dy}{dx} + Py = Qy^n$.

∴ This equation is a Bernoulli's equation. Dividing both the sides by y^6 we get,

$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x}\frac{1}{y^5} = x^2$………… (2).

Take $\frac{1}{y^5} = t$, differentiating w. r. t. x we get, $\frac{-5}{y^6}\frac{dy}{dx} = \frac{dt}{dx}$. ∴ $\frac{1}{y^6}\frac{dy}{dx} = \frac{-1}{5}\frac{dt}{dx}$.

Substituting in (2), we get, $\frac{-1}{5}\frac{dt}{dx} = \frac{1}{x}t = x^2$. Multiplying by -5, we get,

$\frac{dt}{dx} - \frac{5}{x}t = -5x^2$. This is a linear equation of the form $\frac{dt}{dx} + Pt = Q$.

Where $P = -\frac{5}{x}$ and $Q = -5x^2$.

∴ $\mathrm{I.F} = e^{\int P dx} = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5} = \frac{1}{x^5}$

$\therefore$ The solution is $t.(I.F.) = \int Q.(I.F.)\,dx + c$ $\therefore$ $t\,\frac{1}{x^5} = \int \frac{-5}{x^5} x^2 dx + c.$

$\therefore$ $\frac{1}{y^5}\frac{1}{x^5} = -5\int \frac{1}{x^3} dx + c.$ $\therefore$ $\frac{1}{x^5y^5} = -5\left(\frac{-1}{2x^2}\right) + c.$ $\therefore$ $\frac{1}{x^5y^5} = \frac{5}{2x^2} + c.$

This is the required solution.

2. Solve $(y\ \log x - 2\ )y\ dx - x\ dy = 0.$

Solution:

Given $(y\ \log x - 2\ )y\ dx - x\ dy = 0.$ i.e., Given $(y^2 \log x - 2\,y)\ dx - x\ dy$

$\therefore$ $\frac{dy}{dx} = \frac{y^2 \log x - 2y}{x}.$ $\therefore$ $\frac{dy}{dx} + \frac{2y}{x} = \frac{y^2 \log x}{x}$ .... (1).

This is of the form $\frac{dy}{dx} + Py = Qy^n.$

$\therefore$ This equation is a Bernoulli's equation. Dividing both the sides by y^2 we get,

$\frac{1}{y^2}\frac{dy}{dx} + \frac{2}{x}\frac{1}{y} = \frac{\log x}{x}$ .... (2).

Take $\frac{1}{y} = t,$ differentiating w.r.t. x we get

$\frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$ $\therefore$ $\frac{1}{y^2}\frac{dy}{dx} = \frac{-dt}{dx}$

Substituting in (2), we get, $-\frac{dt}{dx} + \frac{2}{x}t = \frac{\log x}{x}$ $\therefore$ $\frac{dt}{dx} - \frac{2}{x}t = -\frac{\log x}{x}.$

This is a linear equation of the form $\frac{dt}{dx} + Pt = Q.$ Where $P = -\frac{2}{x}$ and $Q = -\frac{\log x}{x}.$

$\therefore$ $I.F = e^{\int P\,dx} = e^{\int \frac{-2}{x} dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}.$

$\therefore$ The solution is $t.(I.F.) = \int Q.(I.F.)\,dx + c$

$\therefore$ $t\frac{1}{x^2} = \int -\frac{\log x}{x}.\frac{1}{x^2}\ dx + c.$

$\therefore$ $\frac{1}{y}\frac{1}{x^2} == -\int \log x.\frac{1}{x^3}\ dx + c.$ Using integration by parts, we get,

$\frac{1}{x^2y} = -\left[(\log x)\left(\frac{-1}{2x^2}\right) - \int\left(\frac{-1}{2x^2}\right).\frac{1}{x}\ dx\right] + c.$

$\therefore$ $\frac{1}{x^2y} = \frac{(\log x)}{2x^2} - \frac{1}{2}\int \frac{1}{x^3}\ dx + c.$

$\therefore$ $\frac{1}{x^2y} = \frac{(\log x)}{2x^2} - \frac{1}{2}\left(\frac{-1}{2x^2}\right) + c.$

$\therefore$ $\frac{1}{x^2y} = \frac{(\log x)}{2x^2} + \frac{1}{4x^2} + c.$ This is the required solution.

3. Solve $xy\,(1+x\,y^2)\frac{dy}{dx}=1.$

Solution:

Given $\frac{dy}{dx}=\frac{1}{xy+x^2y^3}$ $\quad\therefore\quad \frac{dx}{dy}=xy\;+\;x^2y^3$ $\quad$ i.e., $\frac{dx}{dy}-yx=y^3x^2$ …This is of the form

$\frac{dx}{dy}+\;Px=Qx^n.$ $\quad\therefore\;$ This equation is a Bernoulli's equation. Dividing both the sides by x^2

we get, $\frac{1}{x^2}\,\frac{dx}{dy}-\frac{y}{x}=y^3$ ……… (2).

Take $\frac{1}{x}=t$, differentiating w. r. t. y we get, $\frac{-1}{x^2}\,\frac{dx}{dy}=\frac{dt}{dy}$ $\quad\therefore\quad \frac{1}{x^2}\,\frac{dx}{dy}=\frac{-dt}{dy}$.

Substituting in (2), we get, $-\frac{dt}{dy}-ty=y^3$ i.e., $\frac{dt}{dy}+yt=-y^3$.

This is a linear equation of the form $\frac{dt}{dy}+Pt=Q$, where $P=y$ and $Q=-y^3$.

$\therefore\quad I.F=e^{\int P\,dy}\;=\;e^{\int y\,dy}\;=\;e^{\frac{y^2}{2}}$ $\quad\therefore\;$ The solution is $t.(I.F.)=\int Q.(I.F.)\,dx+c.$

$\therefore\quad t\,e^{\frac{y^2}{2}}=-\int y^3\,e^{\frac{y^2}{2}}\,dy+c=-\int y^2\,e^{\frac{y^2}{2}}\,y\,dy+c\,.$ Take $\frac{y^2}{2}=u$ $\quad\therefore\quad y\,dy=du.$

$\therefore\quad t\,e^{\frac{y^2}{2}}=-2\int ue^u\,du+c$. Integrating by using Bernoulli's rule, we get,

$te^{\frac{y^2}{2}}=-2(ue^u-e^u)+c.$ $\quad\therefore\quad \frac{1}{x}e^{\frac{y^2}{2}}=-2\left(\frac{y^2}{2}e^{\frac{y^2}{2}}-e^{\frac{y^2}{2}}\right)+c.$

$\therefore\quad \frac{e^{\frac{y^2}{2}}}{x}=-\,y^2\,e^{\frac{y^2}{2}}+2e^{\frac{y^2}{2}}+c.$ $\quad\therefore\quad e^{\frac{y^2}{2}}\left(\frac{1}{x}\;+\;y^2-2\right)=c.$ This is the required solution.

4. Solve $\frac{dy}{dx}=\frac{y}{x-\sqrt{xy}}$.

Solution:

Given $\frac{dx}{dy}=\frac{x-\sqrt{xy}}{y}$ $\quad$ i.e., $\frac{dx}{dy}-\frac{1}{y}x=-\frac{\sqrt{x}}{\sqrt{y}}$ …….. (1).

This is of the form $\frac{dx}{dy}+\;Px=Qx^n.$

$\therefore\;$ This equation is a Bernoulli's equation. Dividing both the sides by $\sqrt{x}$, we get,

$\frac{1}{\sqrt{x}}\,\frac{dx}{dy}-\;\frac{1}{y}\sqrt{x}=-\frac{1}{\sqrt{y}}$……… (2).

Take $\sqrt{x}=t$, differentiating w. r. t. y, we get,

$\frac{1}{2\sqrt{x}}\,\frac{dx}{dy}=\frac{dt}{dy}$ $\quad\therefore\quad \frac{1}{\sqrt{x}}\frac{dx}{dy}=2\,\frac{dt}{dy}$. Substituting in (2), we get,

$2\,\frac{dt}{dy} - \frac{1}{y}t = \frac{-1}{\sqrt{y}}$ i.e., $\frac{dt}{dy} - \frac{1}{2y}t = \frac{-1}{2\sqrt{y}}$.

This is a linear equation of the form $\frac{dt}{dy} + Pt = Q$, where $P = \frac{-1}{2y}$ and $Q = \frac{-1}{2\sqrt{y}}$.

$\therefore$ $I.F = e^{\int P\,dy} = e^{\int \frac{-1}{2y} dy} = e^{-\left(\frac{1}{2}\right)\log y} = \frac{1}{\sqrt{y}}$

$\therefore$ The solution is $t(I.F) = \int Q\,(I.F)\,dy + c.$

$\therefore$ $t\frac{1}{\sqrt{y}} = \int -\frac{1}{2y}\,dy + c = -\frac{1}{2}\log y + c = -\log\sqrt{y} + c$

$\therefore$ $\sqrt{\frac{x}{y}} + \log\sqrt{y} = c$ is the required solution.

5. Solve $r\sin\theta - \cos\theta\frac{dr}{d\theta} = r^2$.

Solution:

Given $\cos\theta\frac{dr}{d\theta} - r\,\sin\theta = -r^2$. Dividing both sides by $\cos\theta$, we get,

$\frac{dr}{d\theta} - r\tan\theta = -r^2\sec\theta$ …… (1).

This is a Bernoulli's equation of the form $\frac{dr}{d\theta} + Pr = Q\,r^n$.

Dividing both the sides by r^2, we get, $\frac{1}{r^2}\frac{dr}{d\theta} - \frac{1}{r}\tan\theta = -\sec\theta$ … … (2)

Take $\frac{1}{r} = t$, differentiating w. r. t. θ, we get, $\therefore$ $\frac{-1}{r^2}\frac{dr}{d\theta} = \frac{dt}{dx}$ $\therefore$ $\frac{1}{r^2}\frac{dr}{d\theta} = \frac{-dt}{dx}$.

Substituting in (2), we get, $-\frac{dt}{dx} - (\tan\theta)t = -\sec\theta$. $\therefore$ $\frac{dt}{dx} + t\tan\theta = \sec\theta$.

This is the linear equation of the form $\frac{dt}{dy} + Pt = Q$, where $P = \tan\theta$ and $Q = \sec\theta$.

$\therefore$ $I.F = e^{\int Pd\theta} = e^{\int \tan\theta d\theta} = e^{\log(\sec\theta)} = \sec\theta$

$\therefore$ The solution is $t(I.F) = \int Q(I.F)d\theta + c.$ $\therefore$ $\frac{1}{r}(\sec\theta) = \int \sec\theta.\sec\theta d\theta + c$

$\therefore$ $\frac{1}{r}(\sec\theta) = \int \sec^2 d\theta + c.$ $\therefore$ $\frac{\sec\theta}{r} = \tan\theta + c$. This is the required solution.

HOME WORK:

1. Solve $\frac{dy}{dx} = y\tan x - y^2\sec x$. **2. Solve $2x\,y' = 10x^3y^5 + y$.**

3. Solve $x(x - y)\,dy + y^2dx = 0$. **4. Solve $\frac{dy}{dx} + \frac{y}{x} = x\,y^2$.**

Exact Differential equation:

A Differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if there exists a function $f(x,y)$ such that $Mdx + Ndy = df$.

Example: The equation $2xy\,dx + x^2\,dy = 0$ is an exact differential equation, since we have

$$2xy\,dx + x^2\,dy = d(x^2 y)$$

Necessary condition:

A necessary condition for the equation $M(x,y)dx + N(x,y)dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

The solution of exact equation is given by

$$\int M(x,y)\,dx \text{ (keeping y as constant)} + \int N(y)dy \text{ (terms of N not containing x)} = C.$$

Problems:

1. Solve $(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0.$

Solution:

Given $(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0$………. (1). i.e., $Mdx + Ndy = 0.$

$\therefore$ $M = 3x^2 + 6xy^2$ and $N = 6x^2y + 4y^3$

$\therefore$ $\frac{\partial M}{\partial y} = 12xy$ and $\frac{\partial N}{\partial x} = 12xy.$ $\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. $\therefore$ The given equation (1) is exact.

$\therefore$ The solution is $\int M(y \text{ is constant})\,dx + \int(\text{terms of N not containing x})dy = c.$

$\therefore$ $\int(3x^2 + 6xy^2)\,dx + \int 4y^3\,dy = c.$

$\therefore$ $x^3 + 3x^2y^2 + y^4 = c.$ This is the required solution.

2. Solve $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)dy = 0.$

Solution:

Given $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)dy = 0$………. (1). i.e., $Mdx + Ndy = 0.$

$\therefore$ $M = x^2 - 4xy - 2y^2$ and $N = y^2 - 4xy - 2x^2$

$\therefore$ $\frac{\partial M}{\partial y} = -4x - 4y$ and $\frac{\partial N}{\partial x} = -4y - 4x.$ $\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\therefore$ The given equation (1) is exact.

$\therefore$ The solution is $\int M(y \text{ is constant})\,dx + \int(\text{terms of N not containing x})dy = c$

$\therefore$ $\int(x^2 - 4xy - 2y^2)\,dx + \int y^2\,dy = c.$

$\therefore$ $\frac{x^3}{3} - \frac{4x^2y}{2} - 2xy^2 + \frac{y^3}{3} = c.$ This is the required solution.

3. Solve $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+[x+\log x-x\sin y]\,dy=0.$

Solution:

Given $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+[x+\log x-x\sin y]\,dy=0$………(1). i.e., $Mdx+Ndy=0.$

$\therefore \quad M=y\left(1+\frac{1}{x}\right)+\cos y \quad$ and $\quad N=x+\log x-x\sin y$

$\therefore \quad \frac{\partial M}{\partial y}=\left(1+\frac{1}{x}\right)-\sin y \quad$ and $\quad \frac{\partial N}{\partial x}=1+\frac{1}{x}-\sin y. \quad \therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

$\therefore$ The given equation (1) is exact.

$\therefore$ The solution is $\int M(y \text{ is constant})\,dx+\int(\text{terms of N not containing x})dy=c.$

$\therefore \quad \int\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\int 0\,dy=c.$

$\therefore$ $y(x+\log x)+x\cos y=c.$ This is the required solution.

4. Solve $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0.$

Solution:

Given equation can be written as $(y\cos x+\sin y+y)dx+(\sin x+x\cos y+x)dy=0$ …. (1).

i.e., $Mdx+Ndy=0.$

$\therefore \quad M=y\cos x+\sin y+y \quad$ and $\quad N=\sin x+x\cos y+x.$

$\therefore \quad \frac{\partial M}{\partial y}=\cos x+\cos y+1 \quad$ and $\quad \frac{\partial N}{\partial x}=\cos x+\cos y+1\,. \quad \therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

$\therefore$ The given equation (1) is exact.

$\therefore$ The solution is $\int M(y \text{ is constant})\,dx+\int(\text{terms of N not containing x})dy=c$

$\therefore \quad \int(y\cos x+\sin y+y)\,dx+\int 0.\,dy=c$

$\therefore$ $y\sin x+x\sin y+xy=c$. This is the required solution.

5. Solve $(2xy+y-\tan y)\,dx+(x^2-x\tan^2 y+\sec^2 y)\,dy=0.$

Solution:

Given $(2xy+y-\tan y)\,dx+(x^2-x\tan^2 y+\sec^2 y)\,dy=0$……….. (1).

i.e., $Mdx+Ndy=0.$

$\therefore \quad M=2xy+y-\tan y \quad$ and $\quad N=x^2-x\tan^2 y+\sec^2 y.$

$\therefore \quad \frac{\partial M}{\partial y}=2x+1-\sec^2 y \quad$ and $\quad \frac{\partial N}{\partial x}=2x-\tan^2 y. \quad \therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

$\therefore$ The given equation (1) is exact.

$\therefore$ The solution is $\int M(\text{y is constant})\,dx + \int(\text{terms of N not containing x})dy = c$

$\therefore$ $\int(2xy + y - \tan y)\,dx + \int \sec^2 y\ dy = c$

$\therefore$ $x^2y + xy - x\tan y + \tan y = c$. This is the required solution.

6. Solve $y\sin 2x\,dx - (1 + y^2 + \cos^2 x)dy = 0$.

Solution:

Given $y\sin 2x\,dx - (1 + y^2 + \cos^2 x)dy = 0$ ……….. (1). i.e., $Mdx + Ndy = 0$.

$\therefore$ $M = y\sin 2x$ and $N = -(1 + y^2 + \cos^2 x)$.

$\therefore$ $\frac{\partial M}{\partial y} = \sin 2x$ and $\frac{\partial N}{\partial x} = -2\cos x\,(-\sin x) = 2\sin x\cos x = \sin 2x$. $\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\therefore$ The given equation (1) is exact.

$\therefore$ The solution is $\int M(\text{y is constant})\,dx + \int(\text{terms of N not containing x})dy = c$.

$\therefore$ $\int(y\sin 2x)\,dx + \int -(1 + y^2)\,dy = c$.

$\therefore$ $y\left(-\frac{\cos 2x}{2}\right) - y - \frac{y^3}{3} = c$. Multiply both sides by -6.

$\therefore$ $3y\cos 2x + 6y + 2y^3 = k$. This is the required solution.

7. Solve $(1 + 2xy\cos x^2 - 2xy)dx + (\sin x^2 - x^2)dy = 0$.

Solution:

Given $(1 + 2xy\cos x^2 - 2xy)dx + (\sin x^2 - x^2)dy = 0$……. (1). i.e., $Mdx + Ndy = 0$.

$\therefore$ $M = 1 + 2xy\cos x^2 - 2xy$ and $N = \sin x^2 - x^2$.

$\therefore$ $\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x$ and $\frac{\partial N}{\partial x} = 2x\cos x^2 - 2x$. $\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\therefore$ The given equation (1) is exact.

$\therefore$ The solution is $\int M(\text{y is constant})\,dx + \int(\text{terms of N not containing x})dy = c$.

$\therefore$ $\int(1 + 2xy\cos x^2 - 2xy)\,dx + \int 0.\,dy = c$.

$\therefore$ $x + y\int 2x\cos x^2 dx - x^2y = c$.

Take $x^2 = t$ $\therefore$ $2x\,dx = dt$

$\therefore$ $x + y\int \cos t\,dt - x^2y = c$. $\therefore$ $x + y\sin t - x^2y = c$.

$\therefore$ $x + y\sin x^2 - x^2y = c$. This is the required solution.

8. Solve $\left(y^2 e^{xy^2} + 4x^3\right) dx + \left(2xye^{xy^2} - 3y^2\right) dy = 0.$

Solution:

Given $\left(y^2 e^{xy^2} + 4x^3\right) dx + \left(2xye^{xy^2} - 3y^2\right) dy = 0 \ldots\ldots\ldots (1).$ i.e., $Mdx + Ndy = 0.$

$\therefore \quad M = y^2 e^{xy^2} + 4x^3$ and $N = 2xye^{xy^2} - 3y^2.$

$\therefore \ \frac{\partial M}{\partial y} = y^2 . e^{xy^2} . 2xy + e^{xy^2} . 2y = 2xy^3 e^{xy^2} + 2y\, e^{xy^2}$ and

$\frac{\partial N}{\partial x} = 2xy\, e^{xy^2} . y^2 + e^{xy^2} . 2y = 2xy^3 e^{xy^2} + 2y\, e^{xy^2}.$

$\therefore \ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. $\therefore$ The given equation (1) is exact.

$\therefore$ The solution is $\int M(\text{y is constant})\, dx + \int (\text{terms of N not containing x}) dy = c.$

$\therefore \ \int \left(y^2 e^{xy^2} + 4x^3\right) dx + \int \left(-3y^2\right) dy = c.$

$\therefore \ y^2 . \frac{e^{xy^2}}{y^2} + x^4 - y^3 = c.$ $\quad \therefore \ e^{xy^2} + x^4 - y^3 = c.$ This is the required solution.

HOME WORK:

1. **Solve $\left(x^4 - 2xy^2 + 2y^4\right) dx - \left(2x^2 y - 4xy^3 + \sin y\right) dy = 0.$**
2. **Solve $y\, e^{xy} dx + \left(xe^{xy} + 2y\right) dy = 0.$**
3. **Solve $\left(\sec x \tan x \tan y - e^x\right) dx + \sec x \sec^2 y\, dy = 0.$**

Equation reducible to exact equation:

A differential equation of the form $Mdx + Ndy = 0$ which is not exact can be reduced to exact equation on multiplying it by an appropriate function is called an integrating factor.

Rule to find Integrating Factors:

Suppose the equation $M\,dx + N\,dy = 0$ is not exact, i.e., $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then we obtain the integrating factors as follows.

(i) If the difference $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is very close to expression of N and $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x say $f(x)$, then I. F is $\mathbf{e}^{\int f(x) dx}$.

(ii) If the difference $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is very close to expression of M and $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of y say $g(y)$, then I. F is $\mathbf{e}^{-\int g(y) dy}$.

Problems:

1. Solve $y(2x-y+1)\,dx + x(3x-4y+3)\,dy = 0$.

Solution:

Given $y(2x-y+1)\,dx + x(3x-4y+3)\,dy = 0$ .... (1)

$\therefore \quad M = 2xy - y^2 + y \quad$ and $\quad N = 3x^2 - 4xy + 3x$.

$\therefore \quad \frac{\partial M}{\partial y} = 2x - 2y + 1 \quad$ and $\quad \frac{\partial N}{\partial x} = 6x - 4y + 3$.

$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore, the equation is not exact. We find that

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -4x + 2y - 2 = -2(2x-y+1)$. It is close to M.

$\therefore \quad \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y(2x-y+1)}.-2(2x-y+1) = -\frac{2}{y} = g(y)$.

$\therefore$ I.F. $= e^{-\int g(y)dy} = e^{-\frac{2}{y}dy} = e^{2\log y} = e^{\log y^2} = y^2$.

Multiplying the equation(1) with y^2, we get,

$(2xy^3 - y^4 + y^3)\,dx + (3x^2y^2 - 4xy^3 + 3xy^2)\,dy = 0$. It is an exact equation.

Now take $M = 2xy^3 - y^4 + y^3 \quad$ and $\quad N = 3x^2y^2 - 4xy^3 + 3xy^2$.

$\therefore$ The solution is $\int M\,dx + \int N(y)\,dy = c$.

$\therefore \int(2xy^3 - y^4 + y^3)\,dx + \int 0\,dy = c$

$\therefore \quad x^2y^3 - xy^4 + xy^3 = c$, is the required solution.

2. Solve $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$

Solution:

Given $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$ ... . (1)

Let $M = (xy^3 + y) \quad$ and $\quad N = 2(x^2y^2 + x + y^4)$

$\therefore \quad \frac{\partial M}{\partial y} = 3xy^2 + 1 \quad$ and $\quad \frac{\partial N}{\partial x} = 2(1 + 2xy^2)$

$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore, the equation is not exact. We find that

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -(xy^2 + 1) \neq 0$. It is close to M.

$\therefore \quad \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y(xy^2+1)}.-(xy^2+1) = -\frac{1}{y} = g(y)$

$\therefore$ $\text{IF} = e^{-\int g(y)\,dy} = e^{-\int -\frac{1}{y}\,dy} = e^{\log y} = y.$

Multiplying the eqution (1) with y, we get,

$(xy^4 + y^2)\,dx + 2\,(x^2\,y^3 + xy + y^5)\,dy = 0$. It is an exact equation

Now take $M = (xy^4 + y^2)$ and $N = 2\,(x^2\,y^3 + xy + y^5)$.

$\therefore$ The solution is $\int M\,dx + \int N(y)\,dy = c.$

$\therefore$ $\int (xy^4 + y^2)\,dx + \int 2y^5\,dy = c.$

$\therefore$ $\mathbf{\frac{x^2y^4}{2} + xy^2 + 2\,\frac{y^6}{6} = c}$, is the required solution.

3. Solve $(y\ \log y)\,dx + (x - \log y)\,dy = 0$

Solution:

Given $(y\ \log y)\,dx + (x - \log y)\,dy = 0 \ldots\ldots\ldots (1)$

Let $M = (y\ \log y)$ and $N = (x - \log y)$.

$\therefore$ $\frac{\partial M}{\partial y} = 1 + \log y$ and $\frac{\partial N}{\partial x} = 1$

$\therefore$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore, the equation is not exact. We find that

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \log y \neq 0$. It is close to M.

$\therefore$ $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y\log y}\,.\log y = \frac{1}{y} = g(y)$

$\therefore$ $\text{I.F.} = e^{-\int g(y)\,dy} = e^{-\int \frac{1}{y}\,dy} = e^{-\log y} = \frac{1}{y}\,.$

Multiplying the equation (1) with $\frac{1}{y}$, we get,

$(\log y)\,dx + \left(\frac{x}{y} - \frac{\log y}{y}\right)\,dy = 0$. It is an exact equation.

Now take $M = \log y$ and $N = \frac{x}{y} - \frac{\log y}{y}$

$\therefore$ The solution is $\int M\,dx + \int N(y)\,dy = c.$

$\therefore$ $\int \log y\,dx + \int -\frac{\log y}{y}\,dy = c.$

$\therefore$ $x\log y - \frac{1}{2}(\log y)^2 = c$, is the required solution.

4. Solve $y(2xy+1)\,dx - x\,dy = 0$.

Solution:

Given $y(2xy+1)\,dx - x\,dy = 0$ ……. (1)

$\therefore\quad M = 2xy^2 + y \quad \text{and} \quad N = -x.$

$\therefore\quad \frac{\partial M}{\partial y} = 4xy + 1 \quad \text{and} \quad \frac{\partial N}{\partial x} = -1.$

$\therefore\quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore, the equation is not exact. We find that

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4xy + 2 = 2(2xy+1)$. It is close to M.

$\therefore\quad \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y(2xy+1)}.2(2xy+1) = \frac{2}{y} = g(y).$

$\therefore\quad I.F. = e^{-\int g(y)dy} = e^{-\int \frac{2}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = \frac{1}{y^2}.$

Multiplying the equation (1) with $\frac{1}{y^2}$, we get,

$\left(2x + \frac{1}{y}\right) dx - \frac{x}{y^2}\, dy = 0$. It is ani exact equation.

Now take $M = 2x + \frac{1}{y}$ and $N = -\frac{x}{y^2}$.

$\therefore$ The solution is $\int M\,dx + \int N(y)\,dy = c.$

$\therefore\quad \int (2xy+1)\frac{1}{y}dx + \int 0\ dy = c. \qquad \therefore\quad \int \left(2x + \frac{1}{y}\right) dx = c.$

$\therefore\quad x^2 + \frac{x}{y} = c$, is the required solution.

5. Solve $y(x+y)\,dx + (x+2y-1)\,dy = 0$.

Solution:

Given $y(x+y)\,dx + (x+2y-1)\,dy = 0$……. (1).

Let $M = y(x+y)$ and $N = (x+2y-1)$.

$\therefore\quad \frac{\partial M}{\partial y} = x + 2y \ \text{and}\ \frac{\partial N}{\partial x} = 1.$

$\therefore\quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore, the equation is not exact. We find that

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = (x+2y-1)$. It is very close to N.

$\therefore\quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{(x+2y-1)}.(x+2y-1) = 1 = f(x)$

$\therefore$ $I.F. = e^{\int f(x)\,dx} = e^{\int 1\,dx} = e^{x}$.

Multiplying the equation(1) with e^x, we get,

$y(x+y)e^x\,dx + (x+2y-1)e^x\,dy = 0$. It is an exact equation

Now take $M = y(x+y)e^x$ and $N = (x+2y-1)e^x$

$\therefore$ The solution is $\int M\,dx + \int N(y)\,dy = c$.

$\therefore$ $\int y(x+y)e^x\,dx + \int 0\,dy = c$. $\therefore$ $\int (yxe^x + \mathbf{y^2}e^x)\,dx = c$.

$\therefore$ $y(xe^x - 1e^x) + y^2e^x = c$, is the required solution.

6. Solve $\left(xy^2 - e^{1/x^3}\right)dx - x^2y\,dy = 0$.

Solution:

Given $\left(xy^2 - e^{1/x^3}\right)dx - x^2y\,dy = 0$ .... (1)

Let $M = xy^2 - e^{1/x^3}$. and $N = -x^2y$.

$\therefore$ $\frac{\partial M}{\partial y} = 2xy$ and $\frac{\partial N}{\partial x} = -2xy$. $\therefore$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore, the equation is not exact.

$\therefore$ $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4xy$. It is very close to N. $\therefore$ $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-4}{x} = f(x)$.

$I.F = e^{\int f(x)dx} = e^{\int \frac{-4}{x}dx} = e^{-4\log x} = \frac{1}{x^4}$.

Multiplying the equation (1) with $\frac{1}{x^4}$, we get,

$\left(xy^2 - e^{1/x^3}\right)\frac{1}{x^4}\,dx - \frac{x^2y}{x^4}\,dy = 0$.

$\therefore$ $\left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}\right)dx - \frac{y}{x^2}\,dy = 0$. It is an exact equation.

Now take $M = \frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}$. $N = -\frac{y}{x^2}$.

$\therefore$ The solution is $\int M\,dx + \int N(y)\,dy = c$.

$\therefore$ $\int \left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}\right)dx + \int 0\,dy = c$. $\therefore$ $y^2\frac{-1}{2x^2} + \frac{1}{3}\int\left(-\frac{3}{x^4}e^{1/x^3}\right)dx = c$.

$\therefore$ $\frac{-y^2}{2x^2} + \frac{1}{3}e^{1/x^3} = c$. $\therefore$ $\frac{e^{-x^3}}{3} - \frac{y^2}{2x^2} = c$ is the required solution.

7. Solve $(6x^2 + 4y^3 + 12y)dx + 3x(1+y^2)dy = 0$.

Solution:

Given $(6x^2 + 4y^3 + 12y)dx + 3x(1+y^2)dy = 0$ ...... (1)

Let $M = (6x^2 + 4y^3 + 12y)$ and $N = 3x(1 + y^2)$. ….(1)

$\therefore \ \frac{\partial M}{\partial y} = 12y^2 + 12$ and $\frac{\partial N}{\partial x} = 3(1 + y^2)$.

$\therefore \ \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore, the equation is not exact. We find that

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 9y^2 + 9 = 9(y^2 + 1) \neq 0$. It is very close to N.

$\therefore \ \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{3x(1+y^2)} \cdot 9(y^2 + 1) = \frac{3}{x} = f(x)$

$\therefore$ IF $= e^{\int f(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\log x} = e^{\log x^3} = x^3$.

Multiplying the equation (1) with, we get,

$(6x^5 + 4x^3y^3 + 12x^3y)dx + 3x^4(1 + y^2)dy = 0$. It is an exact equation

Now take, $M = (6x^5 + 4x^3y^3 + 12x^3y)$ and $N = 3x^4(1 + y^2)$.

$\therefore$ The solution is $\int M\,dx + \int N(y)\,dy = c$.

$\therefore$ The solution is $6\int x^5dx + 4y^3\int x^3dx + 12y\int x^3dx = c$.

$\therefore \ x^6 + x^4y^3 + 3yx^4 = c$ is the required solution.

8. Solve $2xydy - (x^2 + y^2 + 1)dx = 0$.

Solution:

Given $(-x^2 - y^2 - 1)dx + 2xy\,dy = 0$ ….(1).

$\therefore \ M = (-x^2 - y^2 - 1)$ and $N = 2xy$.

$\therefore \ \frac{\partial M}{\partial y} = -2y$ and $\frac{\partial N}{\partial x} = 2y$.

$\therefore \ \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore, the equation is not exact. We find that

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2y - 2y = -4y$. It close to N

$\therefore \ \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}(-4y) = -\frac{2}{x} = f(x)$.

$\therefore$ IF $= e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = \frac{1}{x^2}$.

Multiplying the equation(1) with $\frac{1}{x^2}$, we get,

$\left(-1 - \frac{y^2}{x^2} - \frac{1}{x^2}\right)dx + \frac{2y}{x}\ dy = 0$. It is an exact equation

Now take $M = -1 - \frac{y^2}{x^2} - \frac{1}{x^2}$ and $N = \frac{2y}{x}$.

$\therefore$ The solution is $\int M\,dx + \int N(y)\,dy = c$.

$\therefore \int \left(-1 - \frac{y^2}{x^2} - \frac{1}{x^2}\right) dx + \int 0\ dy = c$.

$\therefore$ $\mathbf{-x + \frac{y^2}{x} + \frac{1}{x} = c}$ is the required solution.

HOME WORK:

1. **Solve** $\mathbf{(x^2 + y^3 + 6x)dx + y^2 x dy = 0}$.

2. **Solve** $\mathbf{(8xy - 9y^2)dx + 2\,(x^2 - 3xy)dy = 0}$.

3. **Solve** $\mathbf{(x^2 + y^2 + x)dx + xydy = 0}$.

4. **Solve** $\mathbf{y(x + y + 1)dx + x(x + 3y + 2)dy = 0}$.

5. **Solve** $\mathbf{(y^4 + 2y)dx + (xy^3 + 2y^2 - 4x)dy = 0}$.

6. **Solve** $\mathbf{(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0}$.

Applications of Differential Equations:

Orthogonal Trajectories (O.T):

Two families of curves are said to be orthogonal trajectories to each other if every member of one family cuts every member of the other family at right angles.

Orthogonal Trajectories in Cartesian form:

Let $f(x, y, a) = 0 \ldots\ldots (1)$, where a is an arbitrary constant be the equation of a family of curves in cartesian form. Then the process of finding the orthogonal trajectories of this family is as follows.

1. Differentiate (1) w. r. t. x and eliminate a to obtain a differential equation of the form $F\left(x,\ y,\ \frac{dy}{dx}\right) = 0$.

2. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to get $F\left(x,\ y,\ -\frac{dx}{dy}\right) = 0$……. (2).

This equation represent a differential equation of the orthogonal trajectories of (1).

3. Solve the equation (2) to obtain a relation of the form $g(x,\ y,\ b) = 0$, where b is an arbitrary constant. This equation represents the orthogonal trajectories of (1).

Problems:

1. Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$, a being arbitrary Constant.

Solution:

Given $y^2 = 4ax$ ……… (1).

Differentiating (1) w. r. t. x, we get, $2y\frac{dy}{dx} = 4a.$

Substituting in (1), We get, $y^2 = 2y\frac{dy}{dx}x.$ $\therefore$ $2y\frac{dy}{dx} = \frac{y^2}{x}.$ $\therefore$ $2x\frac{dy}{dx} - y = 0.$

This is the D. E of the given family (1).

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$. $\therefore$ $2x\left(-\frac{dx}{dy}\right) - y = 0.$ $\therefore$ $2x\frac{dx}{dy} + y = 0.$

Separating the variables, we get, $2x\,dx + y\,dy = 0.$

Integrating we get, $2\int x\,dx + \int y\,dy = c.$ $\therefore$ $x^2 + \frac{y^2}{2} = c.$

$\therefore$ $2x^2 + y^2 = 2c.$ i.e., $2x^2 + y^2 = b.$ This is the required orthogonal trajectories of (1).

2. Find the orthogonal trajectories of the family of coaxial circles $x^2 + y^2 + 2\lambda x + c = 0$, λ being a parameter.

Solution :

Given $x^2 + y^2 + 2\lambda x + c = 0$ ……… (1).

Differentiating (1) w. r. t. x, we get,

$2x + 2y\frac{dy}{dx} + 2\lambda = 0,$ where $y_1 =$. $\therefore$ $\lambda = -\left(x + y\frac{dy}{dx}\right).$ Put in (1).

$\therefore$ $x^2 + y^2 - 2\left(x + y\frac{dy}{dx}\right)x + c = 0.$ $\therefore$ $y^2 - x^2 - 2xy\frac{dy}{dx} + c = 0.$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$. $\therefore$ $y^2 - x^2 + 2xy\frac{dx}{dy} + c = 0.$

$\therefore$ $2xy\frac{dx}{dy} = x^2 - y^2 - c.$ Dividing both sides by y we get, $2x\frac{dx}{dy} = \frac{1}{y}x^2 - y - \frac{c}{y}$

$\therefore$ $2x\frac{dx}{dy} - \frac{1}{y}x^2 = -y - \frac{c}{y}$ Take $x^2 = t.$ Differentiate w.r.t.y. $\therefore$ $2x\frac{dx}{dy} = \frac{dt}{dy}$

$\therefore$ $\frac{dt}{dy} - \frac{1}{y}t = -\left(y + \frac{c}{y}\right)$ This is a linear equation of the form $\frac{dt}{dy} + Pt = Q,$

where $P = -\frac{1}{y}$ and $Q = -\left(y + \frac{c}{y}\right).$ $\therefore$ $I.F = e^{\int P\,dy} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$

$\therefore$ Solution is $t(I.F) = \int Q(I.F)dy + c.$ $\therefore$ $t\frac{1}{y} = \int -\left(y + \frac{c}{y}\right)\frac{1}{y}\,dy + c_1$

$\therefore$ $\frac{x^2}{y} = -\int dy - \int \frac{c}{y^2}\,dy + c_1.$ $\therefore$ $\frac{x^2}{y} = -y + \frac{c}{y} + c_1$. Multiply both sides by y.

$\therefore$ $x^2 = -y^2 + c + c_1 y.$ Put $c_1 = 2\mu.$

$\therefore$ $x^2 + y^2 - 2\mu x - c = 0.$ this is the required orthogonal trajectories of (1).

3. Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, Where λ is a parameter.

Solution:

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ ……… (1).

Differentiating (1) w. r. t. x, we get,

$\frac{2x}{a^2} + \frac{2yy_1}{b^2+\lambda} = 0.$ $\therefore$ $\frac{x}{a^2} = \frac{-yy_1}{b^2+\lambda}$ $\therefore$ $\frac{1}{b^2+\lambda} = \frac{-x}{a^2 yy_1}$.

Substituting in (1), We get,

$\frac{x^2}{a^2} + y^2\left(\frac{-x}{yy_1}\right) = 1.$ $\therefore$ $\frac{x^2}{a^2} - \frac{xy}{a^2 y_1} = 1.$ $\therefore$ $x^2 - \frac{xy}{y_1} = a^2$………. (2).

This is the D. E of the given family (1). Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ i.e., y_1 by $-\frac{1}{y_1}$ in (2).

$\therefore$ $x^2 + xyy_1 = a^2.$ $\therefore$ $x^2 + xy\frac{dy}{dx} = a^2.$ $\therefore$ $xy\frac{dy}{dx} = a^2 - x^2.$

Separating the variables, we get, $y\,dy = \frac{(a^2-x^2)}{x}dx.$

$\therefore$ $y\,dy = \frac{a^2}{x}dx - xdx$ Integrating we get,

$\int ydy = \int \frac{a^2}{x}dx - \int xdx.$ $\therefore$ $\frac{y^2}{2} = a^2\log x - \frac{x^2}{2} + \frac{c}{2}$ $\therefore$ $y^2 = 2a^2\log x - x^2 + c.$

$\therefore$ $x^2 + y^2 - 2a^2\log x - c = 0.$ This is the required orthogonal trajectories of (1).

4. Show that the family of confocal & coaxial parabolas $y^2 = 4a(x + a)$ is self orthogonal.

Solution:

Given $y^2 = 4a(x + a)$ ------ (1)

Differentiating w. r. t. x, we get,

$2y\frac{dy}{dx} = 4a.$ $\therefore$ $a = \frac{yy_1}{2}$, where $y_1 = \frac{dy}{dx}$. Substituting in (1), we get,

$y^2 = 2yy_1\left(x + \frac{yy_1}{2}\right) \quad \therefore \quad y = 2y_1\left(x + \frac{yy_1}{2}\right). \quad \therefore \quad y = 2xy_1 + y$ ------- (2).

This is the D. E. of the given family. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ i.e., y_1 by $-\frac{1}{y_1}$ in (2).

$\therefore \quad y = 2x\left(-\frac{1}{y_1}\right) + y\left(-\frac{1}{y_1}\right)^2.$

$\therefore \quad y = \frac{-2x}{y_1} + \frac{y}{y_1^2}$. Multiply both sides by y_1^2.

$y = 2xy_1 + yy_1^2$ ------ (3).

This is the D.E. of the orthogonal trajectories of (1), which is same as (2).

Hence $y^2 = 4a(x + a)$ is self-orthogonal.

5. Show that the family of conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$, Where λ is a parameter is self orthogonal.

Solution:

Given $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$……… (1). Differentiating w. r. t. x, we get,

$\frac{2x}{a^2+\lambda} + \frac{2yy_1}{b^2+\lambda} = 0. \quad \therefore \quad \frac{x}{a^2+\lambda} = \frac{-yy_1}{b^2+\lambda}$ ……….(2)

We have a property in ratio and proportion that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} = \frac{c}{d} = \frac{a-c}{b-d}$.

$\therefore \quad \frac{x}{a^2+\lambda} = \frac{-yy_1}{b^2+\lambda} = \frac{x+yy_1}{a^2-b^2} \quad \therefore \quad \frac{x}{a^2+\lambda} = \frac{x+yy_1}{a^2-b^2}$ and $\frac{-yy_1}{b^2+\lambda} = \frac{x+yy_1}{a^2-b^2}$

$\therefore \quad \frac{1}{a^2+\lambda} = \frac{x+yy_1}{x(a^2-b^2)}$ and $\frac{1}{b^2+\lambda} = \frac{-(x+yy_1)}{yy_1(a^2-b^2)}$ Substituting in (1) we get,

$x^2\left(\frac{x+yy_1}{x(a^2-b^2)}\right) + y^2\left(\frac{-(x+yy_1)}{yy_1(a^2-b^2)}\right) = 1$

$\therefore \quad \frac{x(x+yy_1)}{(a^2-b^2)} - \frac{y(x+yy_1)}{y_1(a^2-b^2)} = 1$. Multiply both sides by $a^2 - b^2$.

$\therefore \quad x(x + yy_1) - y(x + yy_1) = a^2 - b^2.$

$\therefore \quad (x + yy_1)\left(x - \frac{y}{y_1}\right) = a^2 - b^2$……….(3). This is the D.E of the given family.

Now replacing y_1 by $-\frac{1}{y_1}$ in (3), we get,

$\left(x - \frac{y}{y_1}\right)(x + yy_1) = a^2 - b^2$ ……….(4) This is the D.E of orthogonal trajectories of (1), which is same as (3). Hence the given family of conics is self-orthogonal.

HOME WORK:

1. **Find the orthogonal trajectories of the family of curves $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, Where a is a parameter.**

2. **Find the orthogonal trajectories of the family of coaxial circles $x^2 + y^2 + 2\mu x + c = 0$, μ being the parameter.**

3. **Show that the family of confocal & coaxial parabolas $x^2 = 4a(y + a)$ is self orthogonal.**

4. **Find the orthogonal trajectories of the family of curves $y^2 = cx^3$, Where c is a parameter.**

Orthogonal Trajectories in Polar form:

Let $f(r, \theta, a) = 0 \ldots\ldots (1)$, where a is an arbitrary constant be the equation of a family of curves in polar form. Then the process of finding the orthogonal trajectories of this family is as follows.

1. Differentiate (1) w. r. t. θ and eliminate a to obtain a differential equation of the form

   $f\left(r,\ \theta,\ \frac{dr}{d\theta}\right) = 0.$

2. Replace $\frac{dr}{d\theta}$ by $-\ r^2\ \frac{d\theta}{dr}$ to get $f\left(r,\ \theta,\ -r^2\ \frac{d\theta}{dr}\right) = 0\ldots\ldots$ (2). This equation represent a differential of the orthogonal trajectories of (1).

3. Solve the equation (2) to obtain a relation of the form $g(r,\ \theta,\ b) = 0$, where b is arbitrary Constant. This equation represents the orthogonal trajectories of (1).

Problems:

1. **Find the orthogonal trajectories of the family of cardiods $r = a(1 - \cos\theta)$, where a is the parameter.**

Solution:

Given $r = a(1 + \cos\theta)\ldots\ldots(1)$. Taking Log on both the sides, we get,

$\log r = \log a + \log(1 + \cos\theta)$. Differentiating w. r. t. θ, we get,

$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$. Replacing $\frac{dr}{d\theta}$ by $-\ r^2\ \frac{d\theta}{dr}$, we get,

$\frac{1}{r}\left(-r^2\ \frac{d\theta}{dr}\right) = -\tan(\theta/2)$. $\therefore\ r\frac{d\theta}{dr} = \tan(\theta/2)$. Separating the variables, we get,

$\frac{d\theta}{\tan(\theta/2)} = \frac{dr}{r}$ Integrating, we get, $\int \frac{d\theta}{\tan(\theta/2)} = \int \frac{dr}{r}$ $\therefore \int \frac{dr}{r} = \int \cot(\theta/2)\ d\theta.$

$\therefore\ \log r = 2\log\sin(\theta/2) + \log c$ $\therefore\ \log r - \log\sin^2(\theta/2) = \log c.$

$\therefore \quad \log \frac{r}{\sin^2(\theta/2)} = \log c. \quad \therefore \quad \frac{r}{\sin^2(\theta/2)} = c. \quad \therefore \quad r = c\sin^2(\theta/2).$

$\therefore \quad r = c\,\frac{(1-\cos\theta)}{2} \quad \therefore \quad r = c\,\frac{(1-\cos\theta)}{2} \quad \therefore\ r = b\,(1-\cos\theta).$

This is the required orthogonal trajectories of the family (1).

2. Find the orthogonal trajectories of the family of curves $r = \frac{2a}{(1+\cos\theta)}$, where a is the parameter.

Solution:

Given $r = \frac{2a}{(1+\cos\theta)}$ ……… (1). Take log on both the sides. $\therefore \quad \log r = \log 2a - \log(1+\cos\theta)$.

Differentiating w. r. t. θ, we get, $\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = \tan(\theta/2)$.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$. $\therefore \quad \frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \tan(\theta/2). \quad \therefore \quad -r\frac{d\theta}{dr} = \tan(\theta/2)$.

Separating the variables, we get, $\frac{d\theta}{\tan(\theta/2)} = \frac{-dr}{r}$. Integrating, we get,

$\int \frac{d\theta}{\tan(\theta/2)} = -\int\frac{dr}{r}. \quad \therefore \quad \int\frac{dr}{r} + \int\cot(\theta/2)\ d\theta = 0.$

$\therefore \quad \log r + 2\log\sin(\theta/2) = \log c \quad \therefore \quad \log r + \log\sin^2(\theta/2) = \log c$

$\therefore \quad \log[r\sin^2(\theta/2)] = \log c. \quad \therefore \quad r\sin^2(\theta/2)] = c. \quad \therefore \quad r = \frac{b}{\sin^2(\theta/2)} = \frac{2b}{(1-\cos\theta)}.$

This is the required orthogonal trajectories of the family (1).

3. Find the orthogonal trajectories of the family of curves $r^n = a^n\sin n\theta$, where a is the parameter.

Solution:

Given $r^n = a^n\sin n\theta$ …….. (1)

Take log on both the sides. $\therefore \quad n\log r = n\log a + \log\sin n\theta$.

Differentiating w. r. t. θ, we get, $\frac{n}{r}\frac{dr}{d\theta} = \frac{n\cos n\theta}{\sin n\theta}. \quad \therefore \quad \frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$. $\therefore \quad \frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot n\theta \quad \therefore \quad -r\frac{d\theta}{dr} = \cot n\theta$.

Separating the variables, we get, $\frac{d\theta}{\cot n\theta} = \frac{-dr}{r}$. Integrating, we get,

$\int\frac{d\theta}{\cot n\theta} = -\int\frac{dr}{r}. \quad \therefore \quad \int\frac{dr}{r} + \int\tan n\theta\, d\theta = 0.$

$\therefore \quad \log r + \frac{1}{n}\log\sec n\theta = \log c \quad \therefore \quad n\log r + \log\sec n\theta = n\log c.$

$\therefore \quad \log r^n + \log \sec n\theta = \log c^n$

$\therefore \quad \log(r^n \sec n\theta) = \log c^n$. $\quad \therefore \quad r^n \sec n\theta = c^n$. $\quad \therefore \quad r^n = \frac{b^n}{\sec n\theta}$. Where $b^n = c^n$.

$\therefore \quad r^n = b^n \cos n\theta$. This is the required orthogonal trajectories of the family (1).

4. Find the orthogonal trajectories of the family of curves $r^2 = a^2 \cos 2\theta$, where a is the parameter.

Solution:

Given $r^2 = a^2 \cos 2\theta$ …….. (1).

Take log on both the sides. $\therefore \quad 2\log r = 2\log a + \log \cos 2\theta$.

Differentiating w. r. t. θ, we get, $\frac{2}{r}\frac{dr}{d\theta} = \frac{-2\sin 2\theta}{\cos 2\theta}$. $\quad \therefore \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan 2\theta$.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$. $\quad \therefore \quad \frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan 2\theta$. $\quad \therefore \quad r\frac{d\theta}{dr} = \tan 2\theta$.

Separating the variables, we get, $\frac{d\theta}{\tan 2\theta} = \frac{dr}{r}$. Integrating, we get,

$\int \frac{d\theta}{\tan 2\theta} = \int \frac{dr}{r}$. $\quad \therefore \quad \int \cot 2\theta \, d\theta = \int \frac{dr}{r}$. $\quad \therefore \quad \int \frac{dr}{r} - \int \cot 2\theta \, d\theta = 0$.

$\therefore \quad \log r - \frac{1}{2}\log \sin 2\theta = \log c \quad \therefore \quad 2\log r - \log \sin 2\theta = 2\log c$.

$\therefore \quad \log r^2 - \log \sin 2\theta = \log c^2$. $\quad \therefore \quad \log\left(\frac{r^2}{\sin 2\theta}\right) = \log c^2$. $\quad \therefore \quad \frac{r^2}{\sin 2\theta} = c^2$.

$\therefore \quad r^2 = b^2 \sin 2\theta$. Where $b^2 = c^2$.

This is the required orthogonal trajectories of the family (1).

5. Find the orthogonal trajectories of the family of curves $r^n \cos n\theta = a^n$, where a is the parameter.

Solution:

Given $r^n \cos n\theta = a^n$…….. (1) Take log on both the sides. $\therefore \quad n\log r + \log \cos n\theta = n\log a$.

Differentiating w. r. t. θ, we get, $\frac{n}{r}\frac{dr}{d\theta} + \frac{-n\sin n\theta}{\cos n\theta} = 0$. $\quad \therefore \quad \frac{1}{r}\frac{dr}{d\theta} = \tan n\theta$.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$. $\quad \therefore \quad \frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \tan n\theta \quad \therefore \quad -r\frac{d\theta}{dr} = \tan n\theta$.

Separating the variables, we get, $\frac{d\theta}{\tan n\theta} = \frac{-dr}{r}$. Integrating, we get,

$\int \frac{d\theta}{\tan n\theta} = -\int \frac{dr}{r}$. $\quad \therefore \quad \int \frac{dr}{r} + \int \cot n\theta \, d\theta = 0$.

$\therefore \quad \log r + \frac{1}{n}\log \sin n\theta = \log c \quad \therefore \quad n\log r + \log \sin n\theta = n\log c$.

$\therefore \quad \log r^n + \log \sin n\theta = \log c^n$

$\therefore \; \log(r^n \sin n\theta) = \log c^n$. $\quad \therefore \; r^n \sin n\theta = c^n$. $\quad \therefore \; r^n \sin n\theta = b^n$. Where $b^n = c^n$.

This is the required orthogonal trajectories of the family (1).

HOME WORK:

1. **Find the orthogonal trajectories of the family of cardiods $r = a(1 + \sin\theta)$, where a is the parameter.**
2. **Show that the orthogonal trajectories of the family of curves $r^n = a \sec n\theta$ is the family of curves $r^n = b \operatorname{cosec} n\theta$, where a, b are constants.**
3. **Show that the family of curves $r = 2a(\sin\theta + \cos\theta)$ and $r = 2b(\sin\theta - \cos\theta)$ intersect each other orthogonally.**
4. **Find the orthogonal trajectories of the family of cardiods $\left(r + \frac{k^2}{r}\right)\cos\theta = a$, where a is the parameter.**
5. **Show that the orthogonal trajectories of the family of Cardioids $r = a\cos^2(\theta/2)$ is another family of Cardioids $r = b\sin^2(\theta/2)$.**

Applications of D.E in Newton's law of cooling:

Newton's law of cooling:

The change of temperature of a body is proportional to the difference between the temperature of the body and that of the surrounding medium.

Let T_0 be the temperature of the surrounding medium and T be the temperature of the body at time t, then $\frac{dT}{dt} \propto (T - T_0)$. i.e. $\frac{dT}{dt} = -k(T - T_0)$. Where k is the constant of proportionality. ($-$ sign for cooling down of temperature).

Problems:

1. A body originally at 80℃ cools down to 60℃ in 20 minutes, the temperature of the air being 40℃. What will be the temperature of the body after 40 minutes from the original?

Solution:

Given T = 80°C when t = 0 minute. T = 60°C when t = 20 minutes. T_0 = 40°C.

When t = 40 minutes, T = ?.

By Newton's law of cooling, We have, $\frac{dT}{dt} = -k(T - T_0)$, $k > 0$ is a constant

$\therefore\ \frac{dT}{dt} = -k\,(T-40)$. Separating the variables and integrating, we get,

$\int \frac{dT}{T-40} = -k\int dt$. $\therefore$ $\log(T-40) = -kt + \log c$, where c is a constant.

$\therefore$ $\log(T-40) - \log c = -kt$.

$\therefore$ $\log\left(\frac{T-40}{c}\right) = -kt$. $\therefore$ $\frac{T-40}{c} = e^{-kt}$.

$\therefore$ $T - 40 = ce^{-kt}$ ……. (1).

T = 80°C when t = 0 minute. $\therefore$ 40 = c. $\therefore$ c = 40.

T = 60°C when t = 20 minutes. $\therefore$ $20 = ce^{-20k}$. $\therefore$ $20 = 40e^{-20k}$. $\therefore$ $\frac{20}{40} = \frac{1}{e^{20k}}$

$\therefore$ $e^{20k} = 2$. $\therefore$ 20k = log2. $\therefore$ $k = \frac{1}{20}\log 2$. Substitute in (1).

$\therefore$ $T - 40 = 40e^{-\left(\frac{1}{20}\log 2\right)t}$. $\therefore$ $T = 40 + 40e^{-\left(\frac{1}{20}\log 2\right)t}$. Put t = 40.

$\therefore$ $T = 40 + 40e^{-(2\log 2)} = 40 + 40e^{\log\left(\frac{1}{4}\right)} = 40 + 40 \times \frac{1}{4} = 50^0$c. $\therefore$ $T = 50^0$c.

2. If the temperature of the air 30°C and the substance cools from 100°C to 70°C in 15 minutes, find when the temperature will be 40°C.

Solution:

Given T = 100°C when t = 0 minute. T = 70°C when t = 15 minutes. T_0 = 30°C.

When T = 40°C, t = ?.

By Newton's law of cooling, We have, $\frac{dT}{dt} = -k\,(T - T_0)$, $k > 0$ is a constant

$\therefore\ \frac{dT}{dt} = -k\,(T-30)$. Separating the variables and integrating, we get,

$\int \frac{dT}{T-30} = -k\int dt$. $\therefore$ $\log(T-30) = -kt + \log c$, where c is a constant.

$\therefore$ $\log(T-30) - \log c = -kt$. $\therefore$ $\log\left(\frac{T-30}{c}\right) = -kt$. $\therefore$ $\frac{T-30}{c} = e^{-kt}$.

$\therefore$ $T - 30 = ce^{-kt}$ ……. (1).

T = 100°C when t = 0 minute. $\therefore$ 70 = c. $\therefore$ c = 70.

T = 70°C when t = 15 minutes. $\therefore$ $40 = ce^{-20k}$. $\therefore$ $40 = 70e^{-15k}$. $\therefore$ $\frac{40}{70} = \frac{1}{e^{15k}}$

$\therefore$ $e^{15k} = \frac{70}{40} = \frac{7}{4}$. $\therefore$ $15k = \log\left(\frac{7}{4}\right)$. $\therefore$ $k = \frac{1}{15}\log\left(\frac{7}{4}\right)$. Substitute in (1).

$\therefore$ $T - 30 = 70e^{-\left(\frac{1}{15}\log\frac{7}{4}\right)t}$. Put T = 40°C .

$\therefore\quad 10 = 70e^{-\left(\frac{1}{15}\log\frac{7}{4}\right)t} \quad \therefore\quad e^{-\left(\frac{1}{15}\log\frac{7}{4}\right)t} = \frac{1}{7} \quad \therefore\quad e^{\left(\frac{1}{15}\log\frac{7}{4}\right)t} = 7. \quad \therefore\quad \left(\frac{1}{15}\log\frac{7}{4}\right)t = \log 7.$

$\therefore\quad t = \frac{\log 7}{\frac{1}{15}\log\frac{7}{4}} = 52.17 \approx 52.2$ minutes.

3. If the air is maintained at 30℃ and the temperature of the body cools from 80℃ to 60℃
in 12 minutes. Find the temperature of the body after 24 minutes.

Solution:

Given $T = 80°C$ when $t = 0$ minute. $T = 60°C$ when $t = 12$ minutes. $T_0 = 30°C$.

When $t = 24$ minutes, $T = ?$.

By Newton's law of cooling, We have, $\frac{dT}{dt} = -k\,(T - T_0)$, $k > 0$ is a constant

$\therefore\ \frac{dT}{dt} = -k\,(T - 30)$. Separating the variables and integrating, we get,

$\int \frac{dT}{T-30} = -k\int dt. \quad \therefore\quad \log(T - 30) = -kt + \log c$, where c is a constant.

$\therefore\quad \log(T - 30) - \log c = -kt. \quad \therefore\quad \log\left(\frac{T-30}{c}\right) = -kt. \quad \therefore\quad \frac{T-30}{c} = e^{-kt}.$

$\therefore\quad T - 30 = ce^{-kt} \ldots\ldots (1).$

$T = 80°C$ when $t = 0$ minute. $\therefore\quad 50 = c. \quad \therefore\quad c = 50.$

$T = 60°C$ when $t = 12$ minutes. $\therefore\quad 30 = ce^{-20k}. \quad \therefore\quad 30 = 50e^{-12k}. \quad \therefore\quad \frac{30}{50} = \frac{1}{e^{12k}}$

$\therefore\quad e^{12k} = \frac{50}{30} = \frac{5}{3}. \quad \therefore\quad 12k = \log\left(\frac{5}{3}\right). \quad \therefore\quad k = \frac{1}{12}\log\left(\frac{5}{3}\right)$. Substitute in (1).

$\therefore\quad T - 30 = 50e^{-\left(\frac{1}{12}\log\frac{5}{3}\right)t}. \quad \therefore\quad T = 30 + 50e^{-\left(\frac{1}{12}\log\frac{5}{3}\right)t}.$ Put $t = 24$.

$\therefore\quad T = 30 + 50e^{-\left(\frac{1}{12}\log\frac{5}{3}\right)(24)}. \quad \therefore\quad T = 30 + 50e^{-\left(2\log\frac{5}{3}\right)}. \quad \therefore\quad T = 48°C.$

4. Water at temperature 10□C takes 5 minutes to warm up to 20□C in a room at
temperature 40□C. Find the temperature after 20 minutes.

Solution:

Given $T = 10°C$ when $t = 0$ minute. $T = 20°C$ when $t = 5$ minutes. $T_0 = 40°C$.

When $t = 20$ minutes, $T = ?$.

By Newton's law of cooling, We have, $\frac{dT}{dt} = -k\,(T - T_0)$, $k > 0$ is a constant

$\therefore\ \frac{dT}{dt} = -k\,(T - 40)$. Separating the variables and integrating, we get,

$\int \frac{dT}{T-40} = -k \int dt$. $\therefore$ $\log(T-40) = -kt + \log c$, where c is a constant.

$\therefore$ $\log(T-40) - \log c = -kt$. $\therefore$ $\log\left(\frac{T-40}{c}\right) = -kt$. $\therefore$ $\frac{T-40}{c} = e^{-kt}$.

$\therefore$ $T - 40 = ce^{-kt} \ldots\ldots (1)$.

T = 10°C when t = 0 minute. $\therefore$ c = – 30.

T = 20°C when t = 5 minutes. $\therefore$ $-20 = ce^{-5k}$. $\therefore$ $-20 = -30e^{-5k}$.

$\therefore$ $e^{5k} = \frac{30}{20} = \frac{3}{2}$. $\therefore$ $5k = \log\left(\frac{3}{2}\right)$. $\therefore$ $k = \frac{1}{5}\log\left(\frac{3}{2}\right)$. Substitute in (1).

$\therefore$ $T - 40 = -30e^{-\left(\frac{1}{5}\log\frac{3}{2}\right)t}$. $\therefore$ $T = 40 - 30e^{-\left(\frac{1}{5}\log\frac{3}{2}\right)t}$.   Put t = 20 .

$\therefore$ $T = 40 - 30e^{-\left(\frac{1}{5}\log\frac{3}{2}\right)(20)}$. $\therefore$ $T = 40 - 30e^{-\left(4\log\frac{3}{2}\right)}$. $\therefore$ T = 34.07°C.

5. A bottle of mineral water at a room temperature of 72°F is kept in a refrigerator where thetemperature is 44°F . After half an hour, water cooled to 61°F. How long will it take to cool to 50°F?

Solution:

Given T = 72°F when t = 0 minute. T = 61°F when t = 30 minutes. T_0 =44°F .

When T = 50°F , t = ?.

By Newton's law of cooling, We have, $\frac{dT}{dt} = -k\,(T - T_0)$, $k > 0$ is a constant

$\therefore$ $\frac{dT}{dt} = -k\,(T - 44)$. Separating the variables and integrating, we get,

$\int \frac{dT}{T-44} = -k \int dt$. $\therefore$ $\log(T-44) = -kt + \log c$, where c is a constant.

$\therefore$ $\log(T-44) - \log c = -kt$. $\therefore$ $\log\left(\frac{T-44}{c}\right) = -kt$. $\therefore$ $\frac{T-44}{c} = e^{-kt}$.

$\therefore$ $T - 44 = ce^{-kt} \ldots\ldots (1)$.

T = 72°F when t = 0 minute. $\therefore$ c = 28.

T = 61°F when t = 30 minutes. $\therefore$ $17 = ce^{-30k}$. $\therefore$ $17 = 28e^{-30k}$.

$\therefore$ $e^{30k} = \frac{28}{17}$. $\therefore$ $30k = \log\left(\frac{28}{17}\right)$. $\therefore$ $k = \frac{1}{30}\log\left(\frac{28}{17}\right)$. Substitute in (1).

$\therefore$ $T - 44 = 28e^{-\left(\frac{1}{30}\log\frac{28}{17}\right)t}$.   Put T = 50 .

$\therefore$ $6 = 28e^{-\left(\frac{1}{30}\log\frac{28}{17}\right)t}$ $\therefore$ $e^{-\left(\frac{1}{30}\log\frac{28}{17}\right)t} = \frac{6}{28} = \frac{3}{14}$ $\therefore$ $e^{\left(\frac{1}{30}\log\frac{28}{17}\right)t} = \frac{14}{3}$. $\therefore$ $\left(\frac{1}{30}\log\frac{28}{17}\right)t = \log\frac{14}{3}$.

$\therefore \quad t = \dfrac{\log \frac{14}{3}}{\frac{1}{30}\log\frac{28}{17}} = 92.80$ minutes.

HOME WORK:

1. If a substance cools from $370k$ to $330k$ in 10 minutes, when the temperature of the Surrounding air is $290k$, find the temperature of the substance after 40 minutes.

2. A body in the air at $25℃$ cools from $100℃$ to $75℃$ in one minute, find the temperatureof the body at the end of 3 minutes.

3. A cup of tea at temperature $90℃$ is placed in the room with temperature as $25℃$ and it cools to $60℃$ in 5 minutes, find its temperature after an interval of 5 minutes. Also find the time at which the temperature of tea will come down further by $20℃$.

[Hint: T = 90°C when t = 0 minute. T = 60°C when t = 5 minutes. T_0 = 25°C.
(i) When t = 5 + 5 = 10 minutes, T = ?. (ii) When T = 40°C, t= ?.]
Ans: (i) 43.85°C. (ii) 11.84 min.

Differential equations of first order and higher degree:

A differential equation of the first order of n[th] degree is of the form

$A_0p^n + A_1p^{n-1} + A_2p^{n-2} + \cdots \ldots\ldots + A_n = 0$ ……(1) Where $p = \frac{dy}{dx}$ and $A_0, A_1, A_2, \ldots\ldots A_n$ are functions of x and y. This being a differential equation of first order, the associated general solution will contain only one arbitrary constant.

Equation Solvable for p:

Suppose that the LHS of (1) is expressed as a product of n linear factors, then the equivalent form of (1) is $[p - f_1(x,y)][p - f_2(x,y)] \ldots \ldots\ldots [p - f_n(x,y)] = 0$.

$\therefore$ $[p - f_1(x,y)] = 0$, $[p - f_2(x,y)] = 0$, $\ldots \ldots\ldots$, $[p - f_n(x,y)] = 0$……….. (2)

All these are differential equations of first order and first degree . They can be solved by the known methods.

If $F_1(x,y,c) = 0$, $F_2(x,y,c) = 0$, $\ldots \ldots . (x,y,c) = 0$, respectively represents the solution of equations (2), then the general solution (1) is given by the product of all these solutions. i.e., $F_1(x,y,c).F_2(x,y,c). \ldots \ldots . (x,y,c) = 0$ is the general solution of (1)

Problems:

1. Solve $\left(\frac{dy}{dx}\right)^2 - 7\frac{dy}{dx} + 12 = 0.$

Solution:

Given $p^2 - 7p + 12 = 0$ …….. (1) Where $p = \frac{dy}{dx}$. Factorizing, we get,

$(p - 3)(p - 4) = 0.$ $\therefore$ $p - 3 = 0$ and $p - 4 = 0.$ $\therefore$ $p = 3$ and $p = 4.$

$\therefore$ $\frac{dy}{dx} = 3$ and $\frac{dy}{dx} = 4.$ $\therefore$ $dy = 3dx$ and $dy = 4dx.$ Integrating, we get,

$y = 3x + c$ and $y = 4x + c.$ $\therefore$ $3x - y + c = 0$ and $4x - y + c = 0.$

$\therefore$ The general solution of (1) is $(3x - y + c)(4x - y + c) = 0.$

2. Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$.

Solution:

Given $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$ ……..(1). $\therefore$ $p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$, where $p = \frac{dy}{dx}$ and $\frac{1}{p} = \frac{dx}{dy}$.

$\therefore$ $\frac{p^2-1}{p} = \frac{x^2-y^2}{xy}.$ $\therefore$ $x\,yp^2 - xy = x^2p - y^2p.$ $\therefore$ $x\,yp^2 - x^2p + y^2p - xy = 0.$

$\therefore$ $x\,p(y\,p - x) + y(y\,p - x) = 0.$ $\therefore$ $(y\,p - x)(x\,p + y) = 0.$ $\therefore$ $y\,p - x = 0$ and $x\,p + y = 0.$

$\therefore$ $y\frac{dy}{dx} - x = 0$ and $x\frac{dy}{dx} + y = 0.$ Separating the variables, we get,

$\therefore$ $y\,dy - x\,dx = 0$ and $\frac{1}{y}dy + \frac{1}{x}dx = 0.$ Integrating we get,

$\frac{y^2}{2} - \frac{x^2}{2} = \frac{c}{2}$ and $\log y + \log x = \log c.$ $\therefore$ $y^2 - x^2 = c$ and $\log x\,y = \log c$ i.e., $x\,y = c.$

$\therefore$ $x^2 - y^2 + c = 0$ and $x\,y - c = 0.$

$\therefore$ The general solution of (1) is $(x^2 - y^2 + c)(x\,y - c) = 0.$

3. Solve $p^2 + 2py\cot x = y^2$.

Solution:

Given $p^2 + 2py\cot x = y^2$……… (1). Adding $y^2\cot^2 x$ on both sides we get,

$p^2 + 2py\cot x + y^2\cot^2 x = y^2 + y^2\cot^2 x$. $\therefore$ $(p + y\cot x)^2 = y^2(1+\cot^2 x).$

Taking square root on both sides we get, $p + y\cot x = \pm\, y\,\mathrm{cosec}\, x.$

$\therefore$ $p + y\cot x = y\,\mathrm{cosec}\, x$ and $p + y\cot x = -\,y\,\mathrm{cosec}\, x.$

$\therefore$ p = y cosec x – y cot x and p = – y cosec x – y cot x.

$\therefore$ $\frac{dy}{dx}$ = y (cosec x – cot x) and $\frac{dy}{dx}$ = – y (cosec x + cot x).

$\therefore$ $\frac{dy}{y}$ = (cosec x – cot x)dx and $\frac{dy}{y}$ = – (cosec x + cot x)dx

$\therefore$ log y = log(cosec x – cot x) – log sin x +log c and

log y = – [log(cosec x – cot x) + log sin x] + log c.

$\therefore$ log y = log $\frac{(\text{cosec } x - \cot x)c}{\sin x}$ and log y = log $\frac{c}{(\text{cosec } x - \cot x)\sin x}$

$\therefore$ $y = \frac{(1-\cos x)c}{\sin^2 x} = \frac{(1-\cos x)c}{1-\cos^2 x} = \frac{(1-\cos x)c}{(1-\cos x)(1+\cos x)} = \frac{c}{1+\cos x}$ and $y = \frac{c}{1-\cos x}$.

$\therefore$ y(1 + cos x) = c and y(1−cos x) = c.

$\therefore$ $y(1+\cos x) - c = 0$ and $y(1-\cos x) - c = 0$.

$\therefore$ The general solution of (1) is $[y(1+\cos x) - c][y(1-\cos x) - c]=0$

4. Solve p(p + y) = x(x + y).

Solution:

Given $p^2 + py - x(x+y) = 0$……..(1). This is of the form $ax^2 + bx + c = 0$.

Using $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, we get, $p = \frac{-y\pm\sqrt{y^2+4x^2+4xy}}{2} = \frac{-y\pm\sqrt{(y+2x)^2}}{2} = \frac{-y\pm(y+2x)}{2}$.

$\therefore$ $2p = -y \pm (y+2x)$. $\therefore$ $2p = -y + (y+2x)$ and $\therefore$ $2p = -y - (y+2x)$.

$\therefore$ $2p = 2x$ and $\therefore$ $2p = -2y - 2x$. $\therefore$ $p = x$ and $\therefore$ $p = -y - x$

$\therefore$ $\frac{dy}{dx} = x$ and $\frac{dy}{dx} = -y - x$

$\therefore$ dy = xdx …….(2) and $\frac{dy}{dx} + y = -x$……..(3).

Integrating (2) we get, $y = \frac{x^2}{2} + c$. $\therefore$ $2y = x^2 + 2c$. i.e., $x^2 - 2y + 2c = 0$.

The equation (3) is a Linear Equation of the form $\frac{dy}{dx} + Py = Q$, where P = 1 and $Q = -x$.

$\therefore$ $I.F = e^{\int P dx} = e^{\int 1 dx} = e^x$. $\therefore$ Solution is $y(I.F) = \int Q(I.F)dx + c$.

$\therefore$ $y\,e^x = \int -xe^x dx + c$. $\therefore$ $e^x y = -[xe^x - e^x] + c$.

$e^x y + xe^x - e^x - c = 0$.

$\therefore$ The general solution of (1) is $(x^2 - 2y + 2c)(e^x y + xe^x - e^x - c) = 0$.

5. Solve $xy\left(\frac{dy}{dx}\right)^2 - (x^2+y^2)\frac{dy}{dx} + xy = 0.$

Solution:

Given $xyp^2 - (x^2+y^2)p + xy = 0$.........(1). Using $p = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, we get,

$$p = \frac{(x^2+y^2)\pm\sqrt{(x^2+y^2)^2-4x^2y^2}}{2xy} = \frac{(x^2+y^2)\pm\sqrt{x^4+y^4+2x^2y^2-4x^2y^2}}{2xy} = \frac{(x^2+y^2)\pm\sqrt{x^4+y^4-2x^2y^2}}{2xy}.$$

$$\therefore \quad p = \frac{(x^2+y^2)\pm\sqrt{(x^2-y^2)^2}}{2xy} = \frac{(x^2+y^2)\pm(x^2-y^2)}{2xy}.$$

$\therefore \quad p = \frac{(x^2+y^2)+(x^2-y^2)}{2xy}$ and $p = \frac{(x^2+y^2)-(x^2-y^2)}{2xy}$. $\therefore \quad p = \frac{2x^2}{2xy}$ and $p = \frac{2y^2}{2xy}$.

$\therefore \quad p = \frac{x}{y}$ and $p = \frac{y}{x}$. $\therefore \quad \frac{dy}{dx} = \frac{x}{y}$ and $\frac{dy}{dx} = \frac{y}{x}$. Separating the variables, we get,

$y\,dy = x\,dx$ and $\frac{dy}{y} = \frac{dx}{x}$. Integrating we get,

$\frac{y^2}{2} = \frac{x^2}{2} + \frac{c}{2}$ and $\log y = \log x + \log c$

$\therefore \quad y^2 = x^2 + c$ and $\log y = \log cx$ i.e., $y = cx$.

$\therefore \quad x^2 - y^2 + c = 0$ and $y - cx = 0$.

$\therefore$ The general solution of (1) is $(x^2 - y^2 + c)(y - cx) = 0$.

HOME WORK:

1. Solve $x^2p^2 + xp - (y^2 + y) = 0.$

2. Solve $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0.$

3. Solve $p^2 - 2p\sinh x - 1 = 0.$

General solution of an ordinary differential equation:

A solution of an ordinary differential equation is an expression for the dependent variable in **terms of the independent one(s) which satisfies the relation.**

The general solution of a differential equation includes all possible solutions and typically includes arbitrary constants.

Particular solution of an ordinary differential equation:

A solution of a differential equation that contains no arbitrary constants is called a particular solution.

Singular solution of an ordinary differential equation:

A singular solution of a differential equation is also a type of particular solution, but it cannot be taken from the general solution by designating the values of the random constant.

i.e., A singular solution is a solution not obtainable by assigning particular values to the arbitrary constants of the general solution.

Clairaut's Equation:

An equation of the form $y = P x + f(p)$ is known as Clairaut's equation.

General and Singular solution of Clairaut's equation:

1. The general solution of the Clairaut's equation $y = P x + f(p)$ is obtained on replacing p by c.

   i.e., $y = cx + f(c)$ ………(1) is the general solution.

2. The singular solution of the Clairaut's equation $y = P x + f(p)$ is obtained by differentiating the

   general solution (1) partially w. r. t. c and eliminating c from it.

Problems:

1. Solve $y = x p + \frac{a}{p}$. Also find the singular solution.

Solution:

Given $y = p x + \frac{a}{p}$ ……….(1). This is a Clairaut's equation form $y = p x + f(p)$.

Put $p = c$. $\therefore$ $y = c x + \frac{a}{c}$ …….(2). Which is the required general solution of (1).

Differentiate (2) w. r. t. c.

$\therefore$ $0 = x + a\left(\frac{-1}{c^2}\right)$. $\therefore$ $x = \frac{a}{c^2}$. $\therefore$ $c^2 = \frac{a}{x}$. $\therefore$ $c = \sqrt{\frac{a}{x}}$. Substitute in (2).

$\therefore$ $y = \sqrt{\frac{a}{x}}\, x + a\sqrt{\frac{x}{a}}$. $\therefore$ $y = \sqrt{ax} + \sqrt{ax}$. $\therefore$ $y = 2\sqrt{ax}$. $\therefore$ $y^2 = 4ax$.

Which is the required singular solution of (1).

2. Solve $P = \log (p x – y)$. Also find the singular solution.

Solution:

Given $P = \log(p x - y)$. ∴ $px - y = e^p$. ∴ $y = px - e^p$……(1).

This is a Clairaut's equation form $y = p x + f(p)$. Put $p = c$.

∴ $y = cx - e^c$…….(2). Which is the required general solution of (1).

Differentiate (2) w. r. t. c. ∴ $0 = x - e^c$. ∴ $e^c = x$. ∴ $c = \log x$. Substitute in (2).

∴ $y = x \log x - x$. Which is the required singular solution of (1).

3. Solve $p = \sin(y - x p)$. Also find its singular solution.

Solution:

Given $p = \sin(y - x p)$. ∴ $\sin^{-1} p = y - x p$. ∴ $y = p x + \sin^{-1} p$……….(1).

This is a Clairaut's equation of the form $y = p x + f(p)$. Put $p = c$.

∴ $y = c x + \sin^{-1} c$. Which is the required general solution of (1).

Differentiate (2) w. r. t. c. ∴ $0 = x + \frac{1}{\sqrt{1-c^2}}$. ∴ $x = -\frac{1}{\sqrt{1-c^2}}$. ∴ $x^2 = \frac{1}{1-c^2}$.

∴ $1 - c^2 = \frac{1}{x^2}$. ∴ $c^2 = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$. ∴ $c = \frac{\sqrt{x^2-1}}{x}$. Substitute in (2).

∴ $y = \frac{\sqrt{x^2-1}}{x} x + \sin^{-1} \frac{\sqrt{x^2-1}}{x^2}$. ∴ $y = \sqrt{x^2 - 1} + \sin^{-1} \frac{\sqrt{x^2-1}}{x^2}$.

Which is the required singular solution of (1).

4. Find the general and singular solutions of $x p^2 + p x - p y + 1 - y = 0$.

Solution:

Given $x p^2 + p x - p y + 1 - y = 0$. ∴ $x p^2 + p x + 1 = p y + y$

∴ $x p (p + 1) + 1 = y (p + 1)$. Dividing both sides by $p + 1$, we get,

$x p + \frac{1}{p+1} = y$. ∴ $y = p x + \frac{1}{p+1}$ ………(1).

This is a Clairaut's equation of the form $y = p x + f(p)$. Put $p = c$.

∴ $y = c x + \frac{1}{c+1}$ …….. (2). Which is the required general solution of (1).

Differentiate (2) w. r. t. c. ∴ $0 = x - \frac{1}{(c+1)^2}$. ∴ $x = \frac{1}{(c+1)^2}$. ∴ $(c + 1)^2 = \frac{1}{x}$.

∴ $c + 1 = \frac{1}{\sqrt{x}}$. ∴ $c = \frac{1}{\sqrt{x}} - 1$. Substitute in (2).

∴ $y = \left(\frac{1}{\sqrt{x}} - 1\right) x + \sqrt{x}$. ∴ $y = \sqrt{x} - x + \sqrt{x}$. ∴ $y = 2\sqrt{x} - x$.

Which is the required singular solution of (1).

HOME WORK:

1. Find the general and singular solutions of $y = px - \sqrt{1 + p^2}$.

2. Find the general and singular solutions of p = cos y cos px + sin y sin px.

Equations Reducible to Clairaut's Equations:

1. Solve the equation (p x – y) (p y + x) = 2p by reducing into Clairaut's form by taking the Substitutions $X = x^2$, $Y = y^2$

Solution:

Given (p x –y) (p y + x) = 2p ……….(1).

Take $X = x^2$ and $Y = y^2$. ∴ d X =2 x d x and d Y = 2y d y.

Now take $P = \frac{dY}{dX} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x}p$. ∴ $p = \frac{x}{y}P$. Substituting in (1), we get,

$\left(\frac{x}{y}Px - y\right)\left(\frac{x}{y}Py + x\right) = 2\frac{x}{y}P$. ∴ $\left(\frac{x^2}{y}P - y\right)(xP + x) = 2\frac{x}{y}P$.

∴ $\left(\frac{x^2P - y^2}{y}\right)x(P + 1) = 2\frac{x}{y}P$. Multiplying both sides by $\frac{y}{x}$ we get,

$(x^2P - y^2)(P + 1) = 2P$. ∴ $(XP - Y)(P + 1) = 2P$. ∴ $(XP - Y) = \frac{2P}{(P+1)}$.

∴ $Y = PX - \frac{2P}{P+1}$.

This is a Clairaut's equation of the form y = p x + f (p). Put P = c.

∴ $Y = cX - \frac{2c}{c+1}$. ∴ $y^2 = cx^2 - \frac{2c}{c+1}$. Which is the required general solution of (1).

2. Solve the equation $(p - 1)e^{3x} + p^3e^{2y} = 0$ by reducing into Clairaut's form, taking the substitutions $u = e^x$, $v = e^y$.

Solution:

Given $(p - 1)e^{3x} + p^3e^{2y} = 0$ …………(1).

Take $u = e^x$ and $v = e^y$. ∴ $du = e^x dx$ and $dv = e^y dy$.

Now take $P = \frac{dv}{du} = \frac{e^y dy}{e^x dx} = \frac{v}{u}p$. ∴ $p = \frac{u}{v}P$. Substituting in (1), we get,

$\left(\frac{u}{v}P - 1\right)u^3 + \frac{u^3}{v^3}P^3v^2 = 0$. ∴ $\left(\frac{uP - v}{v}\right)u^3 + \frac{u^3}{v}P^3 = 0$.

Multiplying both sides by $\frac{v}{u^3}$ we get, $uP - v + P^3 = 0$.

∴ $v = Pu + P^3$. This is a Clairaut's equation of the form y = p x + f (p). Put P = c.

∴ $v = cu + c^3$. ∴ $e^y = c\,e^x + c^3$. Which is the required general solution of (1).

3. Solve the equation (p x – y) (py + x) = a^2p by reducing into Clairaut's form, taking the substitutions $X = x^2$, $Y = y^2$.

Solution:

Given (p x –y) (p y + x) = a^2 p …………(1).

Take $X = x^2$ and $Y = y^2$. ∴ d X =2 x d x and d Y = 2y d y.

Now take $P = \frac{dY}{dX} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x}$ p. ∴ $p = \frac{x}{y}$ P. Substituting in (1), we get,

$\left(\frac{x}{y}\text{ P x } - y\right)\left(\frac{x}{y}\text{ P y } + x\right) = a^2\frac{x}{y}$ P. ∴ $\left(\frac{x^2}{y}\text{ P } - y\right)(x\text{ P } + x) = a^2\frac{x}{y}$ P.

∴ $\left(\frac{x^2\text{P}-y^2}{y}\right)x(\text{P } + 1) = a^2\frac{x}{y}$ P. Multiplying both sides by $\frac{y}{x}$ we get,

$(x^2\text{P} - y^2)(\text{P } + 1) = a^2$ P. ∴ $(X\text{P} - \text{Y})(\text{P } + 1) = a^2$P. ∴ $(X\text{P} - \text{Y}) = \frac{a^2\text{ P}}{(\text{P } + 1)}$.

∴ $Y = P\,X - \frac{a^2 P}{P + 1}$. This is a Clairaut's equation of the form y = p x + f (p). Put P = c.

∴ $Y = c\,X - \frac{a^2 c}{c + 1}$. ∴ $y^2 = c\,x^2 - \frac{a^2 c}{c + 1}$. Which is the required general solution of (1).

4. Solve sin y $\cos^2$x = $\cos^2$y. p^2 + sin x cos x cos y. p. Take u = sin x and v = sin y.

Solution:

Given sin y $\cos^2$x = $\cos^2$y. p^2 + sin x cos x cos y. p……….(1)

Take u = sin x and v = sin y. ∴ du = cos x dx and dv = sin y dy.

Now take $P = \frac{dv}{du} = \frac{\cos y\,dy}{\cos x\,dx} = \frac{\cos y}{\cos x}$ p. ∴ $p = \frac{\cos x}{\cos y}$ P. Substituting in (1), we get,

sin y $\cos^2$x = $\cos^2$y. $\frac{\cos^2 x}{\cos^2 y}$ P^2+ sin x cos x cos y. $\frac{\cos x}{\cos y}$. P

∴ sin y $\cos^2$x = $\cos^2$x. P^2+ sin x $\cos^2$x. P. Dividing both sides by $\cos^2$x, we get,

sin y = P^2+ sin x. P. ∴ v = P^2+ u P. ∴ v = P u + P^2.

This is a Clairaut's equation of the form y = p x + f (p). Put P = c

∴ v = c u + c^2. ∴ sin y = c sin x + c^2. This is the required general solution of (1).

HOME WORK:

1. Solve the equation x^2(y – px) = p^2y by reducing into Clairaut's form, using $X = x^2$, $Y = y^2$.

2. Solve the equation (p – 1)$e^{4x} + p^2 e^{2y} = 0$ by reducing into Clairaut's form, using u = e^{2x},
v = e^{2y}.