NAGARJUNA COLLEGE OF ENGINEERING AND TECHNOLOGY

(An Autonomous Institution under VTU)

DEPARTMENT OF MATHEMATICS

CLASS NOTES FOR I SEM B.E.

2023-24

Module-2

Series Expansion and Multivariable Calculus

Syllabus:

Taylor's and Maclaurin's series expansion for one variable (Statement only) – problems. Indeterminate forms-L'Hospital's rule. Problems. Partial differentiation, total derivative-differentiation of composite functions. Jacobian and problems. Maxima and minima for a function of two variables- Problems.

Taylor's series Expansion:

The Taylor's series of a function f(x) about a point x = a is given by

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f'^{v}(a) + \ldots\ldots\ldots\ldots$$

OR

$$y(x) = y(a) + \frac{(x-a)}{1!} y'(a) + \frac{(x-a)^2}{2!} y''(a) + \frac{(x-a)^3}{3!} y'''(a) + \frac{(x-a)^4}{4!} y'^{v}(a) + \cdots\ldots$$

Problems:

1. **Expand $\log_e x$ in powers of $(x-1)$ and hence evaluate $\log(1.1)$ correct to four decimal places using Taylor's series Expansion.**

Solution:

Taylor's series is given by

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f'^{v}(a) + \ldots\ldots \quad (1)$$

By data $f(x) = log$; $\quad a = 1. \quad f(1) = \log_e 1 = 0.$

$f'(x) = \frac{1}{x} \quad \therefore f(1) = 1; \quad f''(x) = -\frac{1}{x^2} \quad \therefore f''(1) = -1,$

$f'''(x) = \frac{2}{x^3} \quad \therefore f'''(1) = 2; \quad f'^{v}(x) = -\frac{6}{x^4} \quad \therefore f'^{v}(1) = -6$

Taylor's series (1) *with* $a = 1$ *is given by*

$$\log_e x = 0 + (x-1)1 + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{6}(2) + \frac{(x-1)^4}{24}(-6) + \cdots$$

$$\therefore \quad \log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots\ldots\ldots\ldots$$

$Now\ substitute\ x = 1.1\ to\ obtain\ \log(1.1)$

$\log_e(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} - \frac{(1.1-1)^4}{4} + \dots\dots\dots$

$Thus\ \log_e(1.1) = 0.0953.$

2. Find the Taylor's series expansion of $f(x) = \log \cos x$ at $x = \frac{\pi}{3}$ up to fourth degree term.

Solution:

Taylor's series is given by

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f'^{v}(a) + \dots\dots$$

$$\therefore\ f(x) = f\left(\frac{\pi}{3}\right) + \left(x-\frac{\pi}{3}\right)f'\left(\frac{\pi}{3}\right) + \frac{\left(x-\frac{\pi}{3}\right)^2}{2!}f''\left(\frac{\pi}{3}\right) + \frac{\left(x-\frac{\pi}{3}\right)^3}{3!}f'''\left(\frac{\pi}{3}\right) + \frac{\left(x-\frac{\pi}{3}\right)^4}{4!}f'^{v}\left(\frac{\pi}{3}\right) + \text{--}(1)$$

Given $f(x) = \log(\cos x)$; $a = \frac{\pi}{3}$, $\therefore\ f\left(\frac{\pi}{3}\right) = \log\left(1/2\right) = -\log 2$

Differentiating f(x)successively we get,

$f'(x) = \frac{1}{\cos x}.(-\sin x) = -\tan x \qquad \therefore\ f'\left(\frac{\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3};$

$f''(x) = -\sec^2 x \qquad \therefore\ f''\left(\frac{\pi}{3}\right) = -\sec^2\frac{\pi}{3} = -4$

$f'''(x) = (-2\sec x)\sec x \tan x = -2\sec^2 x \tan x;$

so that $f'''\left(\frac{\pi}{3}\right) = -2\sec^2\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) = -8\sqrt{3}$

$f'^{v}(x) = -2[\sec^2 x.\sec^2 x + \tan x.2\sec x.\tan x] = -2\sec^4 x - 4\sec^2 x\tan^2 x;$

so that $f'^{v}\left(\frac{\pi}{3}\right) = -2\sec^4\left(\frac{\pi}{3}\right) - 4\sec^2\left(\frac{\pi}{3}\right)\tan^2\left(\frac{\pi}{3}\right) = -80$. Substituting in (1) we get,

$$\therefore\ \log(\cos x) = -\log 2 - \sqrt{3}\left(x-\frac{\pi}{3}\right) + (-4)\frac{\left(x-\frac{\pi}{3}\right)^2}{2!} + (-8\sqrt{3})\frac{\left(x-\frac{\pi}{3}\right)^3}{3!} + (-80)\frac{\left(x-\frac{\pi}{3}\right)^4}{3!} + ..$$

$$\therefore\ \log(\cos x) = -\log 2 - \sqrt{3}\left(x-\frac{\pi}{3}\right) - 2\left(x-\frac{\pi}{3}\right)^2 - \frac{4}{\sqrt{3}}\left(x-\frac{\pi}{3}\right)^3 - \frac{10}{3}\left(x-\frac{\pi}{3}\right)^4 + \cdots$$

3. Expand $\sin x$ in powers of $\left(x-\frac{\pi}{2}\right)$ up to the term containing $\left(x-\frac{\pi}{2}\right)^4$. Hence find the value of $\sin 91^0$ correct to 4 decimal places.

Solution:

Using Taylor's series we have,

$$f(x) = f\left(\frac{\pi}{2}\right) + \left(x - \frac{\pi}{2}\right) f'\left(\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} f''\left(\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!} f'''\left(\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^4}{3!} f'^{V}\left(\frac{\pi}{2}\right) + \ldots (1)$$

Given $f(x) = \sin x$, $\therefore\ f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$ $f'(x) = \cos x$, $\therefore\ f'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

$f''(x) = -\sin x$, $\therefore\ f''\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1.$

$f'''(x) = -\cos x$, $\therefore\ f'''\left(\frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right) = 0.$

.$f'^{V}(x) = \sin x$, $\therefore\ f'^{V}\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$ Substituting in (1) we get,

$$\therefore\ \sin x = 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!} + \cdots$$

To determine $\sin 91^0$, take $x = 91^0$ so that $x - \frac{\pi}{2} = 1° = \frac{\pi}{180}$ radian

$$\therefore\ \sin 91^0 \approx 1 - \frac{1}{2}\left(\frac{\pi}{180}\right)^2 + \frac{1}{24}\left(\frac{\pi}{180}\right)^4 \approx 0.9998$$

4. Expand $\tan^{-1} x$ in powers of $(x - 1)$ up to the term containing fourth degree.

Solution:

Taylor's series is given by

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f'^{V}(a) + \ldots\ldots$$

$$\therefore\ f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \frac{(x-1)^4}{4!} f'^{V}(1) + \text{--}(1)$$

Given $f(x) = \tan^{-1} x$; $a = 1$, $\therefore\ f(1) = \tan^{-1} 1 = \frac{\pi}{4}$

Differentiating f(x)successively we get,

$f'(x) = \frac{1}{1+x^2}$ $\therefore\ f'(1) = \frac{1}{2}$;

$(1 + x^2)\ f'(x) = 1$

$(1 + x^2)f''(x) + 2x\ f'(x) = 0$ $\therefore\ f''(1) = -\frac{1}{2}$

$(1 + x^2)\ f'''(x) + 2x\ f''(x) + 2x\, f''(x) + 2\, f'(x) = 0$

$(1 + x^2)\ f'''(x) + 4x\ f''(x) + 2\, f'(x) = 0$

so that $f'''(1) = \frac{1}{2}$

$(1 + x^2)f'^{V}(x) + 6x\, f'''(x) + 6\, f''(x) =$

so that $f'^{v}(1) = 0$

Substituting in (1) we get,

$$\therefore\ \tan^{-1}x = \frac{\pi}{4} + (x-1)\frac{1}{2} + \frac{(x-1)^2}{2!}\left(-\frac{1}{2}\right) + \frac{(x-1)^3}{3!}\left(\frac{1}{2}\right) + \frac{(x-1)^4}{4!}(0) + \text{--}$$

$$\therefore\ \tan^{-1}x = \frac{\pi}{4} + (x-1)\frac{1}{2} + \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12}$$

HOME WORK

1. **Expand $\tan x$ in powers of $\left(x - \frac{\pi}{4}\right)$ upto 3^{rd} degree term.**
2. **Expand $\sqrt{x}$ in powers of $(x-2)$ up to the third degree terms.**

Maclaurin's series Expansion:

Put a=0 in Taylor's series expansion, then we obtain

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \ldots \ldots \ldots \ldots$$

This expression is called Maclaurin's series Expansion.

Alternatively, it can also be written as

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \cdots \ldots \ldots \ldots \ldots$$

Problems:

1. Using Maclaurin's series, expand $y = \sqrt{(1+\sin 2x)}$ in powers of x up to the term containing x^4.

Solution:

Given $1 + \sin 2x = (\cos^2 x + \sin^2 x) + 2\sin x\cos x = (\cos x + \sin x)^2$.

$\sqrt{(1+\sin 2x)} = \cos x + \sin x = f(x)$, we find that $f(0) = 1$.

$f'(x) = -\sin x + \cos x$, $\therefore\ f'(0) = 1$. $f''(x) = -\cos x - \sin x$, $\therefore\ f''(0) = -1$.

$f'''(x) = \sin x - \cos x$, $\therefore\ f'''(0) = -1$. $f'^{v}(x) = \cos x + \sin x$, $\therefore\ f'^{v}(0) = 1$.

Maclaurin's series is given by

$$f(x) = f(0) + (x)f'(0) + \frac{(x)^2}{2!}f''(0) + \frac{(x)^3}{3!}f'''(0) + \frac{(x)^4}{4!}f'^{v}(0) + \cdots$$

$$\therefore\ \sqrt{(1+\sin 2x)} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

2. Find the Maclaurin's series of $\log(1+x)$.

Hence deduce that $\log\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots$

Solution:

Let $y = \log(1+x)$, $\therefore\ y(0) = \log 1 = 0$

$y_1 = \frac{1}{1+x}$, $\therefore\ y_1(0) = 1$; $y_2 = -\frac{1}{(1+x)^2}$, $\therefore\ y_2(0) = -1$

$y_3 = \frac{2}{(1+x)^3}$, $\therefore\ y_3(0) = 2$; $y_4 = -\frac{6}{(1+x)^4}$, $\therefore\ y_4(0) = -6$

$y_5 = \frac{24}{(1+x)^5}$, $\therefore\ y_5(0) = 24$.

Maclaurin's series is given by

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \frac{x^5}{5!}y_5(0) + \cdots \ldots\ldots\ldots\ldots$$

$$\therefore\ \log(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \frac{x^5}{5!}(24) + \ldots\ldots\ldots\ldots$$

$$\therefore\ \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \qquad \ldots\ldots..\ \text{(i)}$$

changing x to $-x$ in the above expression, we get

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots \qquad \ldots\ldots\ldots\ \text{(ii)}$$

$$\therefore\ \log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = \frac{1}{2}[\log(1+x) - \log(1-x)]$$

$$\therefore\ \log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right\} + \frac{1}{2}\left\{x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} \ldots\right\} \text{ by using (i)and (ii).}$$

$$\therefore\ \log\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

3. Prove that $e^{\sin x} = 1 + x + \frac{x^2}{2!} - 3\frac{x^4}{4!} - 8\frac{x^5}{5!} + \cdots$

Solution:

Let $y = e^{\sin x}$, $\therefore\ y(0) = e^{\sin 0}$ $\therefore\ y(0) = e^0 = 1$;

Differentiating we get,

$y_1 = e^{\sin x}\cos x = y\cos x$ $\therefore\ y_1(0) = 1$. $y_2 = y_1\cos x - y\sin x$. $\therefore\ y_2(0) = 1$.

$y_3 = y_2\cos x - 2y_1\sin x - y\cos x \quad \therefore \quad y_3(0) = 0.$

$y_4 = y_3\cos x - 3y_2\sin x - 3y_1\cos x + y\sin x \quad \therefore \quad y_4(0) = -3.$

$y_5 = y_4\cos x - 4y_3\sin x - 6y_2\cos x + 4y_1\sin x + y\cos x \quad \therefore \quad y_5(0) = -8.$

Maclaurin's series is given by

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \frac{x^5}{5!}y_5(0) + \cdots \ldots\ldots\ldots\ldots$$

$$\therefore\ e^{\sin x} = 1 + x + \frac{x^2}{2!} - 3\frac{x^4}{4!} - 8\frac{x^5}{5!} + \cdots$$

4. Prove that $e^{x\sin x} = 1 + x^2 + \frac{x^4}{3} + \ldots\ldots\ldots$

Solution:

Let $y = e^{x\sin x} \quad \therefore \quad y(0) = 1$

Differentiating we get,

$y_1 = e^{x\sin x}(x\cos x + \sin x) \quad \therefore \quad y_1(0) = 0$

$y_1 = y(x\cos x + \sin x)$

$y_2 = y(-x\sin x + 2\cos x) + y_1(x\cos x + \sin x)$

$y_2(0) = 1(0 + 2) + 0 \quad \therefore \quad y_2(0) = 2$

$y_3 = y(-x\cos x - 3\sin x) + y_1(-x\sin x + 2\cos x) + y_1(-x\sin x + 2\cos x)$
$\qquad + y_2(x\cos x + \sin x)$

$y_3 = y(-x\cos x - 3\sin x) + 2y_1(-x\sin x + 2\cos x) + y_2(x\cos x + \sin x)$

$y_3(0) = 0 + 0 + 0 \quad \therefore \quad y_3(0) = 0$

$y_4 = y(x\sin x - 4\cos x) + y_1(-x\cos x - 3\sin x) + 2y_1(-x\cos x - 3\sin x)$
$\qquad + 2y_2(-x\cos x + \sin x) + y_2(-x\sin x + 2\cos x) + y_3(x\cos x + \sin x)$

$y_4(0) = -4 + 0 + 0 + 8 + 4 \quad \therefore \quad y_4(0) = 8.$

Maclaurin's series is given by

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \frac{x^5}{5!}y_5(0) + \ldots\ldots\ldots\ldots$$

$$\therefore\ e^{x\sin x} = 1 + x(0) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(8) + \ldots\ldots\ldots\ldots$$

$$\therefore\ e^{x\sin x} = 1 + x^2 + \frac{x^4}{3} + \ldots\ldots\ldots\ldots$$

HOME WORK

1. Prove that log (1 + sin x) = $x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \text{--------------------}$
2. Using Maclaurin's series, expand $y = e^x \cos x$ in powers of x.
3. Using Maclaurin's series, expand $\log(\sec x)$ in the powers of x up to the term containing x^5
4. Using Maclaurin's series, expand $y = e^{a \sin^{-1} x}$ in powers of x
5. Expand $\log(1 + e^x)$ in power of x upto the term containing x^4
6. Using Maclaurin's series, expand $y = \log(1 + \cos x)$ in powers of x.
7. Using Maclaurin's series, expand $y = e^{\tan^{-1} x}$ in powers of x
8. Using Maclaurin's series, expand $y = \log(\sec x + \tan x)$ in powers of x.

Indeterminate forms:

Expressions of the forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, $0^0, \infty^0, 1^\infty$ are called the indeterminate forms.

L' Hospital's Rule:

Statement: If $f(x)$ and $g(x)$ are two functions such that

(i) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ i.e., $f(a) = 0 = g(a)$.

(ii) $f'(x)$ and $g'(x)$ exist and $g'(a) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Again, if $f'(a) = 0 = g'(a)$ then, we have $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$ and soon.

Note: $\lim_{x\to 0} \frac{\sin x}{x} = 1$, $\lim_{x\to 0} \frac{\tan x}{x} = 1$, $\lim_{x\to 0} (1 + x)^{\frac{1}{x}} = e$.

Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$

Problems:

1. Evaluate $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$.

Solution:

Let $L = \lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$. $\left(\frac{0}{0}\right)$. Applying the L'Hospital's Rule, we get,

$L = \lim\limits_{x\to 0} \frac{xe^x + e^x - \frac{1}{1+x}}{2x}$ $\left(\frac{0}{0}\right)$. Again, applying the L'Hospital's Rule, we get,

$\therefore$ $L = \lim\limits_{x\to 0} \frac{xe^x + e^x + e^x + \frac{1}{(1+x)^2}}{2}$. $\therefore$ $L = \frac{3}{2}$

2. Evaluate $\lim\limits_{x\to 1} \frac{x^x - x}{x - 1 - \log x}$.

Solution:

Let $L = \lim\limits_{x\to 1} \frac{x^x - x}{x - 1 - \log x}$ $\left(\frac{0}{0}\right)$. Applying the L'Hospital's Rule, we get,

$L = \lim\limits_{x\to 1} \frac{x^x(1+\log x) - 1}{1 - \frac{1}{x}}$ $\left(\frac{0}{0}\right)$. Using $\frac{d(x^x)}{dx} = x^x(1 + \log x)$.

Again using L'Hospital's Rule, we get,

$L = \lim\limits_{x\to 1} \frac{x^x(1+\log x)^2 + x^x\left(\frac{1}{x}\right)}{1 + \frac{1}{x^2}} = \frac{1+1}{1}$ $\therefore$ $L = 2$.

3. Evaluate $\lim\limits_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$

Solution:

Let $L = \lim\limits_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$ $\left(\frac{0}{0}\right)$. Applying the L'Hospital's Rule, we get,

$L = \lim\limits_{x\to 0} \frac{e^x + e^{-x} - 2\left(\frac{1}{1+x}\right)}{x\cos x + \sin x}$ $\left(\frac{0}{0}\right)$. Again, applying the L'Hospital's Rule, we get,

$L = \lim\limits_{x\to 0} \frac{e^x - e^{-x} + 2\left(\frac{1}{(1+x)^2}\right)}{x(-\sin x) + \cos x + \cos x} = \frac{1-1+2}{0+1+1} = 1$.

4. Evaluate $\lim\limits_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$

Solution:

Let $L = \lim\limits_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$ $\left(\frac{0}{0}\right)$. Multiplying and dividing denominator by x, we get,

$L = \lim\limits_{x\to 0} \frac{\tan x - x}{x^2 \frac{\tan x}{x} x} = \lim\limits_{x\to 0} \frac{\tan x - x}{x^3} . \lim\limits_{x\to 0} \frac{x}{\tan x} = \lim\limits_{x\to 0} \frac{\tan x - x}{x^3} . 1$ $\left(\frac{0}{0}\right)$. Using $\lim\limits_{x\to 0} \frac{\tan x}{x} = 1$.

Applying the L'Hospital's Rule, we get,

$L = \lim\limits_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim\limits_{x\to 0} \frac{\tan^2 x}{3x^2} = \frac{1}{3} \lim\limits_{x\to 0} \left(\frac{\tan x}{x}\right)^2 = \frac{1}{3} . 1 = \frac{1}{3}$. Using $\sec^2 x - 1 = \tan^2 x$.

Note:

In case of indeterminate forms $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, we first reduce them to $\frac{0}{0}$ form.

5. Evaluate $\lim\limits_{x\to 0} \frac{\log x}{\cot x}$

Solution:

Let L=$\lim\limits_{x\to 0} \frac{\log x}{\cot x}$ $\left(\frac{\infty}{\infty}\right)$

Applying the L'Hospital's Rule, we get,

L=$\lim\limits_{x\to 0} \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x} = -\lim\limits_{x\to 0} \frac{\sin^2 x}{x}$ $\left(\frac{0}{0}\right)$. Again, applying the L'Hospital's Rule, we get,

L=$-\lim\limits_{x\to 0} \frac{2\sin x \cos x}{1} = 0$.

6. Evaluate $\lim\limits_{x\to 0} \log_{\sin x} \sin 2x.$

Solution:

Let L=$\lim\limits_{x\to 0} \log_{\sin x} \sin 2x$=$\lim\limits_{x\to 0} \frac{\log \sin 2x}{\log \sin x}$ $\left(\frac{-\infty}{-\infty}\right)$. Using $\log_b a = \frac{\log a}{\log b}$

Applying the L'Hospital's Rule, we get,

L=$\lim\limits_{x\to 0} \frac{2\left(\frac{\cos 2x}{\sin 2x}\right)}{\frac{\cos x}{\sin x}}$=$\lim\limits_{x\to 0} \frac{2\cot 2x}{\cot x} = \lim\limits_{x\to 0} 2.\frac{\tan x}{\tan 2x}$ $\left(\frac{0}{0}\right)$. Again, applying the L'Hospital's Rule, we get,

L=$\lim\limits_{x\to 0} \frac{2\sec^2 x}{2\sec^2 2x} = \frac{2}{2} = 1$

7. Evaluate $\lim\limits_{x\to 1} (1 - x^2) \tan\left(\frac{\pi x}{2}\right)$

Solution:

Let L=$\lim\limits_{x\to 1} (1 - x^2) \tan\left(\frac{\pi x}{2}\right)$ $(0 \times \infty)$

∴ L=$\lim\limits_{x\to 1} \frac{(1-x^2)}{\cot\left(\frac{\pi x}{2}\right)}$ $\left(\frac{0}{0}\right)$. Applying the L'Hospital's Rule, we get,

L=$\lim\limits_{x\to 1} \frac{-2x}{-\operatorname{cosec}^2\left(\frac{\pi x}{2}\right)\frac{\pi}{2}} = \frac{2}{\frac{\pi}{2}} = \frac{4}{\pi}$.

8. Evaluate $\lim\limits_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

Solution:

Let L=$\lim\limits_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$. $(\infty - \infty)$.

$\therefore \quad L=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}\ \left(\frac{0}{0}\right)$. Applying the L'Hospital's Rule, we get,

$L=\lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x}\ \left(\frac{0}{0}\right)$. Again, applying the L'Hospital's Rule, we get,

$L=\lim_{x\to 0}\frac{\sin x}{-x\sin x+2\cos x}=0$

9. Evaluate $\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{1}{log(x-1)}\right)$.

Solution:

Let $L=\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{1}{log(x-1)}\right)\quad(\infty-\infty)$

$\therefore \quad L=\lim_{x\to 2}\left[\frac{\log(x-1)-(x-2)}{(x-2)\log(x-1)}\right]\quad\left(\frac{0}{0}\right)$

Applying the L'Hospital's Rule, we get,

$L=\lim_{x\to 2}\left\{\frac{\frac{1}{(x-1)}-1}{\frac{(x-2)}{(x-1)}+\log(x-1)}\right\}=\lim_{x\to 2}\left\{\frac{2-x}{(x-2)+(x-1)\log(x-1)}\right\}\quad\left(\frac{0}{0}\right)$.

Again, applying the L'Hospital's Rule, we get,

$L=\lim_{x\to 2}\frac{-1}{1+\log(x-1)+(x-1).\frac{1}{(x-1)}}=\frac{-1}{2}$.

10. Evaluate $\lim_{x\to 0}\left[\frac{1}{x^2}-cot^2x\right]$

Solution:

Let $L=\lim_{x\to 0}\left[\frac{1}{x^2}-cot^2x\right]\qquad(\infty-\infty)$

$=\lim_{x\to 0}\left[\frac{1}{x^2}-\frac{1}{tan^2x}\right]=\lim_{x\to 0}\left[\frac{tan^2x-x^2}{x^2tan^2x}\right]$ Multiplying and dividing denominator by x^2, we get,

$L=\lim_{x\to 0}\left[\frac{tan^2x-x^2}{x^2\frac{tan^2x}{x^2}x^2}\right]=\lim_{x\to 0}\frac{tan^2x-x^2}{x^4}.\lim_{x\to 0}\left(\frac{x}{\tan x}\right)^2=\lim_{x\to 0}\frac{tan^2x-x^2}{x^4}.1\ \left(\frac{0}{0}\right)$. Using $\lim_{x\to 0}\frac{\tan x}{x}=1$.

Applying the L'Hospital's Rule, we get,

$L=\lim_{x\to 0}\frac{2\tan x sec^2x-2x}{4x^3}\ \left(\frac{0}{0}\right)$. Again, applying the L'Hospital's Rule, we get,

$L=\lim_{x\to 0}\frac{2\tan x.2sec^2x\tan x+2sec^4x-2}{12x^2}=\lim_{x\to 0}\frac{4sec^2x\,tan^2x+2sec^4x-2}{12x^2}=\frac{2}{12}\lim_{x\to 0}\frac{2sec^2x\,tan^2x+sec^4x-1}{x^2}$

Again, applying the L'Hospital's Rule, we get,

Now $\sec^4x-1=(\sec^2x-1)(\sec^2x+1)=\tan^2x(\sec^2x+1)=\sec^2x\tan^2x+\tan^2x$

$\therefore\ L = \frac{1}{6}\lim_{x\to 0}\frac{2sec^2x\ tan^2x + sec^2x\ tan^2x + tan^2x}{x^2} = \frac{1}{6}\lim_{x\to 0}\frac{3sec^2x\ tan^2x + tan^2x}{x^2} = \frac{1}{6}\lim_{x\to 0}\frac{tan^2x(3sec^2x+1)}{x^2}$

$\therefore\ L = \frac{1}{6}\left\{\lim_{x\to 0}\left[\frac{\tan x}{x}\right]^2 . \lim_{x\to 0}(3sec^2x + 1)\right\} = \frac{1}{6}(1)(4) = \frac{2}{3}.$

Indeterminate forms: 0^0, ∞^0, 1^∞.

Let $L=\lim_{x\to a}[f(x)]^{g(x)}$

Taking logarithm on both sides, we get,

$\log_e L = \lim_{x\to a} g(x)\log[f(x)]$. Evaluate the limit on RHS by using L'Hospital's rule to get,

$\log_e L = k \quad (say)$. Thus $L = e^k$.

1. Evaluate $\lim_{x\to 1} x^{\frac{1}{1-x}}$

Solution:

Let $L=\lim_{x\to 1} x^{\frac{1}{1-x}}$ (1^∞) Taking log on both sides, we get,

$\log_e L = \lim_{x\to 1}\frac{1}{1-x}\log x = \lim_{x\to 1}\frac{\log x}{1-x}$ $\left(\frac{0}{0}\right)$. Applying the L'Hospital's Rule, we get,

$\log_e L = \lim_{x\to 1}\frac{\frac{1}{x}}{-1} = -1.$ $\quad \therefore\ L = e^{-1} = \frac{1}{e}$

2. Evaluate lim $\lim_{x\to\frac{\pi}{2}} (sin\ x)^{tan\ x}$ $\quad (1^\infty)$

Solution:

Let $L=\lim_{x\to\frac{\pi}{2}} sin\ x^{tan\ x}$. Taking log on both sides, we get,

$\log_e L = \lim_{x\to\frac{\pi}{2}} \tan x \log sinx \quad (0\times\infty)$

$\log_e L = \lim_{x\to\frac{\pi}{2}} \frac{\log \sin x}{\cot x}$ $\left(\frac{0}{0}\right)$. Applying the L'Hospital's Rule, we get,

$\log_e L = \lim_{x\to\frac{\pi}{2}}\frac{\left(\frac{1}{\sin x}\right)\cos x}{-cosec^2x} = \lim_{x\to\frac{\pi}{2}}\frac{\frac{\cos x}{\sin x}}{-\frac{1}{sin^2x}} = -\lim_{x\to\frac{\pi}{2}} \sin x\cos x.$

$\therefore\ \log_e L = 0. \quad \therefore\ L = e^0 = 1.$

3. Evaluate $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$ $\quad (1^\infty)$

Solution:

Let L = $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$. Taking log on both sides, we get,

$$\log_e L = \lim_{x\to 0}\frac{1}{x}\log\left(\frac{a^x+b^x+c^x}{3}\right) = \lim_{x\to 0}\frac{\log[a^x+b^x+c^x]-\log 3}{x} \quad \left(\frac{0}{0}\right)$$

Applying the L'Hospital's Rule, we get,

$$\log_e L = \lim_{x\to 0}\frac{\frac{1}{a^x+b^x+c^x}[a^x\log a+b^x\log b+c^x\log c]}{1}$$

$\therefore\ \log_e L = \frac{1}{3}[\log a+\log b+\log c] = \frac{1}{3}\log(abc) = log(abc)^{\frac{1}{3}}$

$\therefore\ L = (abc)^{\frac{1}{3}}$

4. Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ (1^∞)

Solution:

Let L= $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$. Taking log on both sides, we get,

$\log_e L = \lim_{x\to 0}\frac{1}{x^2}log\left(\frac{\tan x}{x}\right)$ $\left(\frac{0}{0}\right)$. Applying the L'Hospital's Rule, we get,

$$\log_e L = \lim_{x\to 0}\frac{\frac{1}{\left(\frac{\tan x}{x}\right)}\cdot\left(\frac{x.sec^2x-\tan x}{x^2}\right)}{2x} = \lim_{x\to 0}\frac{x}{\tan x}.\lim_{x\to 0}\frac{x.sec^2x-\tan x}{2x^3} \quad \left(\frac{0}{0}\right)$$

$\therefore\ \log_e L = 1.\lim_{x\to 0}\frac{x(2sec^2x\tan x)+sec^2x-sec^2x}{6x^2} = \lim_{x\to 0}\frac{x(2sec^2x\tan x)}{6x^2} = \frac{1}{3}\lim_{x\to 0}\frac{sec^2x\ \tan x}{x}$

$\therefore\ \log_e \mathrm{L} = \frac{1}{3}\lim_{x\to 0}\sec^2 x.\lim_{x\to 0}\frac{\tan x}{x} = \frac{1}{3}$. $\quad\therefore\ \mathrm{L} = e^{\frac{1}{3}}$.

5. Evaluate $\lim_{x\to 0} x^{\sin x}$

Solution:

Let L= $\lim_{x\to 0} x^{\sin x}$ (0^0). Taking log on both sides, we get,

$\log_e L = \lim_{x\to 0}\sin x\log x \quad (0\times -\infty)$

$\therefore\ L=\lim_{x\to 0}\frac{\log x}{cosec\, x}$ $\left(\frac{-\infty}{-\infty}\right)$. Applying the L'Hospital's Rule, we get,

$$\log_e L = \lim_{x\to 0}\frac{\frac{1}{x}}{-cosec\, x.\cot x} = -\lim_{x\to 0}\frac{\sin x}{x}.\lim_{x\to 0}\tan x = (-1).0 = 0.$$

L= $e^0 = 1$.

6. Evaluate $\lim\limits_{x\to 0} (cot\ x)^{tan\ x}$

Solution:

Let L= $\lim\limits_{x\to 0} (cot\ x)^{tan\ x}$ (∞^0). Taking log on both sides, we get,

$\log_e L = \lim\limits_{x\to 0} \tan x \log \cot x.$ $\quad (0 \times -\infty)$.

$\therefore$ $\log_e L = \lim\limits_{x\to 0} \frac{\log \cot x}{\cot x}$ $\left(\frac{-\infty}{-\infty}\right)$. Applying the L'Hospital's Rule, we get,

$\log_e L = \lim\limits_{x\to 0} \frac{-cosec^2 x}{-\cot x . cosec^2 x} = \lim\limits_{x\to 0} \tan x = 0.$ $\quad \therefore$ L = $e^0 = 1$.

7. Evaluate $\lim\limits_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$

Solution:

Let L= $\lim\limits_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$ $\left(\frac{0}{0}\right)$. Because $\lim\limits_{x\to 0} (\mathbf{1} + \boldsymbol{x})^{\frac{1}{x}} = e$

Applying the L'Hospital's Rule, we get,

L= $\lim\limits_{x\to 0} \frac{\frac{d}{dx}\left[(1+x)^{\frac{1}{x}}\right]}{1} = \lim\limits_{x\to 0} \frac{du}{dx}$ ….(1)

Where $u = (1+x)^{\frac{1}{x}}$. Taking logarithm on both sides, we get,

$\log u = \frac{1}{x}\log(1+x) = \frac{\log(1+x)}{x}$. Using $\log(1+\mathrm{x}) = \mathrm{x} - \frac{\mathrm{x}^2}{2} + \frac{\mathrm{x}^3}{3} - \frac{\mathrm{x}^4}{4} + \frac{\mathrm{x}^5}{5} - \cdots$, we get,

$\log u == \frac{1}{x}\left[\mathrm{x} - \frac{\mathrm{x}^2}{2} + \frac{\mathrm{x}^3}{3} - \frac{\mathrm{x}^4}{4} + \frac{\mathrm{x}^5}{5} - \cdots\right] = 1 - \frac{\mathrm{x}}{2} + \frac{\mathrm{x}^2}{3} - \frac{\mathrm{x}^3}{4} + \frac{\mathrm{x}^4}{5} - \cdots$

Differentiating w. r. t. x, we get,

$\frac{1}{u}\frac{du}{dx} = -\frac{1}{2} + \frac{2\mathrm{x}}{3} - \frac{3\mathrm{x}^2}{4} + \frac{4\mathrm{x}^3}{5} - \cdots$

$\therefore$ $\frac{du}{dx} = u\left[-\frac{1}{2} + \frac{2\mathrm{x}}{3} - \frac{3\mathrm{x}^2}{4} + \frac{4\mathrm{x}^3}{5} - \cdots\right] = (1+x)^{\frac{1}{x}}\left[-\frac{1}{2} + \frac{2\mathrm{x}}{3} - \frac{3\mathrm{x}^2}{4} + \frac{4\mathrm{x}^3}{5} - \cdots\right]$

$\therefore$ $\lim\limits_{x\to 0} \frac{du}{dx} = \lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} . \lim\limits_{x\to 0} \left[-\frac{1}{2} + \frac{2\mathrm{x}}{3} - \frac{3\mathrm{x}^2}{4} + \frac{4\mathrm{x}^3}{5} - \cdots\right] = \mathrm{e}\left(-\frac{1}{2}\right) = \frac{-e}{2}$. Substitute in (1).

$\therefore$ L = $\frac{-e}{2}$.

8. Find the values of a, b, c such that $\lim\limits_{x\to 0} \frac{x(a+b\cos x) - c\ sinx}{x^5} = 1$

Solution:

Given $\lim\limits_{x\to 0} \frac{x(a+b\cos x) - c\ sinx}{x^5} = 1$ …… (1). We observe that the LHS is of the form$\left(\frac{0}{0}\right)$.

∴ Applying the L'Hospital's Rule on LHS, we get,

$\lim_{x\to 0} \frac{x(-b\sin x)+(a+b\cos x)-c\cos x}{5x^4} = 1$ … (2). Applying limit on LHS, we get,

$\frac{a+b-c}{0} = 1.$ ∴ a + b −c = 0 … (3).

As the denominator on LHS of (2) is zero for $x \to 0$, we assume that LHS of (2) is of the form $\left(\frac{0}{0}\right)$ and applying the L'Hospital's Rule on LHS, we get,

$\lim_{x\to 0} \frac{-b(x\cos x+\sin x)-b\sin x+c\sin x}{20x^3} = 1.$ Again applying the L'Hospital's Rule on LHS, we get,

$\lim_{x\to 0} \frac{-b(-x\sin x+\cos x+\cos x)-b\cos x+c\cos x}{60x^2} = 1.$

∴ $\lim_{x\to 0} \frac{bx\sin x+\cos x(-3b+c)}{60x^2} = 1$ ……. (4). By applying limit on LHS, we get,

$\frac{-3b+c}{0} = 1.$ ∴ $c = 3b$….. (5) Substituting in(4) we get,

$\lim_{x\to 0} \frac{bx\sin x}{60x^2} = 1.$ ∴ $\lim_{x\to 0} \frac{b\sin x}{60x} = 1.$ ∴ $\frac{b}{60} \lim_{x\to 0} \frac{\sin x}{x} = 1.$ Using $\lim_{x\to 0} \frac{\sin x}{x} = 1$, we get

$\frac{b}{60} = 1.$ ∴ $b = 60.$ Put in (5). ∴ $c = 3(60) = 180.$ Substitute in (3).

∴ a + 60 −180 = 0. ∴ a =120.

∴ $a = 120,\quad b = 60,\quad c = 180.$

HOME WORK:

1. Evaluate the following limits:

(i) $\lim_{x\to 0} \frac{a^x-b^x}{x}$ **(ii)** $\lim_{x\to 0} \frac{x\cos x-\sin x}{x^2\sin x}$ **(iii)** $\lim_{x\to \frac{\pi}{2}} \frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}$ **(iv)** $\lim_{x\to 0} \left[\frac{a}{x} - \cot\left(x/a\right)\right]$

(v) $\lim_{x\to 0} \frac{x^2+2\cos x-2}{x\sin^3 x}$ **(vi)** $\lim_{x\to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right]$ **(vii)** $\lim_{x\to a} \left[2-\left(\frac{x}{a}\right)\right]^{\tan\left(\frac{\pi x}{2a}\right)}$

(viii) $\lim_{x\to 1} (1-x^2)^{\frac{1}{\log(1-x)}}$ **(ix)** $\lim_{x\to \infty} \left(1+\frac{a}{x}\right)^x$

2. Find the constants a, b, c such that $\lim_{x\to 0} \frac{ae^x-b\cos x+ce^{-x}}{x\sin x}$ **may be equal 2.**

PARTIAL DIFFERENTIATION:

Definition:

Consider a function $u = f(x, y)$ of two independent variables x and y . The derivative of $f(x, y)$ w. r. t. x by treating y as a constant is called the partial derivative of $u=f(x, y)$ w. r. t. x. It is denoted as $\frac{\partial f}{\partial x}$ or f_x or $\frac{\partial u}{\partial x}$ or u_x .

Thus $\frac{\partial f}{\partial x} = f_x = \lim_{h\to 0}\frac{f(x+h\,, y)-f(x\,,y)}{h}$

Similarly, the partial derivative of $u = f(x, y)$ w. r. t. y by treating x as a constant is called the partial derivative of $u=f(x, y)$ w. r. t. y. It is denoted as $\frac{\partial f}{\partial y}$ or f_y or $\frac{\partial u}{\partial y}$ or u_y. Thus $\frac{\partial f}{\partial y} = f_y = \lim_{k\to 0}\frac{f(x\,, y+k)-f(x\,,y)}{k}$

The second order Partial derivatives of $u = f(x, y)$ are denoted by

(i) $f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$, (ii) $f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$,

(iii) $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}$, (iv) $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$

Also note that $\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$ i.e., $f_{xy} = f_{yx}$

Problems:

1. Find the first and second order partial derivatives of $z = x^3 + y^3 - 3axy$.

Solution:

Given $z = x^3 + y^3 - 3axy$ ....(1)

Differentiating (1) partially w. r. t. x (treating y as constant) we get,

$\frac{\partial z}{\partial x} = 3x^2 - 3ay$ .....(2)

Differentiating (1) partially w. r. t. y (treating x as constant) we get,

$\frac{\partial z}{\partial y} = 3y^2 - 3ax$ ... ... (3)

Differentiating (2) partially w. r. t. x we get, $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = 6x$

Differentiating (3) partially w. r. t. y we get, $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = 6y$

Differentiating (3) partially w. r. t. x we get, $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -3a$ .....(4)

Differentiating (2) partially w. r. t. y we get, $\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = -3a$ ......(5)

∴ From (4) and (5) we get, $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$

2. If $(x+y)z = x^2 + y^2$, prove that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

Solution:

Given $z = \frac{x^2+y^2}{x+y}$ …….. (1)

Differentiating (1) partially w. r. t. x (treating y as constant) we get,

$$\frac{\partial z}{\partial x} = \frac{(x+y)2x-(x^2+y^2)(1)}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

Differentiating (1) partially w. r. t. y (treating x as constant) we get,

$$\frac{\partial z}{\partial y} = \frac{(x+y)2y-(x^2+y^2)(1)}{(x+y)^2} = \frac{y^2+2xy-x^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2} == \frac{x^2+2xy-y^2-y^2-2xy+x^2}{(x+y)^2} = \frac{2x^2-2y^2}{(x+y)^2}$$

$$\therefore \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2(x^2-y^2)}{(x+y)^2} = \frac{2(x+y)(x-y)}{(x+y)^2} = \frac{2(x-y)}{(x+y)}$$

$$\therefore \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{4(x-y)^2}{(x+y)^2} \;\ldots..(2)$$

Now $4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4\left[1 - \frac{(x^2+2xy-y^2)}{(x+y)^2} - \frac{(y^2+2xy-x^2)}{(x+y)^2}\right]$

$$= 4\left[\frac{(x+y)^2-(x^2+2xy-y^2)-(y^2+2xy-x^2)}{(x+y)^2}\right]$$

$$=4\left[\frac{x^2+2xy+y^2-x^2-2xy+y^2-y^2-2xy+x^2}{(x+y)^2}\right] = 4\left[\frac{x^2-2xy+y^2}{(x+y)^2}\right]$$

$$\therefore \quad 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = \frac{4(x-y)^2}{(x+y)^2} \;\ldots..(3)$$

From (2) and (3) we get $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

3. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$ show that $\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}$

Solution:

Given $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$..........(1)

Differentiating (1) partially w. r. t. x (treating y as constant) we get,

$$\frac{\partial u}{\partial x} = x^2 \frac{1}{\left(1+\frac{y^2}{x^2}\right)}\left(-\frac{y}{x^2}\right) + \tan^{-1}\left(\frac{y}{x}\right)(2x) - y^2 \frac{1}{\left(1+\frac{x^2}{y^2}\right)}\left(\frac{1}{y}\right)$$

$$= \frac{-x^2 y}{x^2+y^2} - \frac{y^3}{y^2+x^2} + 2x\tan^{-1}\left(\frac{y}{x}\right) = \frac{-y(x^2+y^2)}{x^2+y^2} + 2x.\tan^{-1}\left(\frac{y}{x}\right)$$

$\therefore \frac{\partial u}{\partial x} = -y + 2x.\tan^{-1}\left(\frac{y}{x}\right)$ ....(2)

Differentiating (1) partially w. r. t. y (treating x as constant) we get,

$$\frac{\partial u}{\partial y} = x^2 \frac{1}{\left(1+\frac{y^2}{x^2}\right)}\left(\frac{1}{x}\right) - \{y^2 \frac{1}{\left(1+\frac{x^2}{y^2}\right)}\left(-\frac{x}{y^2}\right) + \tan^{-1}\left(\frac{x}{y}\right).(2y)\}$$

$\therefore \frac{\partial u}{\partial y} = \frac{x^3}{x^2+y^2} + \frac{xy^2}{y^2+x^2} - 2y\tan^{-1}\left(\frac{x}{y}\right) = \frac{x(x^2+y^2)}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right)$

$\therefore \frac{\partial u}{\partial y} = x - 2y.\tan^{-1}\left(\frac{x}{y}\right)$ ....(3)

Differentiating (3) partially w. r. t. x we get,

$$\frac{\partial^2 u}{\partial x\,\partial y} = 1 - 2y\frac{1}{\left(1+\frac{x^2}{y^2}\right)}\left(\frac{1}{y}\right) = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} \quad ....(4)$$

Differentiating (2) partially w. r. t. y we get,

$$\frac{\partial^2 u}{\partial y\,\partial x} = -1 + 2x.\frac{1}{\left(1+\frac{y^2}{x^2}\right)}\left(\frac{1}{x}\right) = -1 + \frac{2x^2}{y^2+x^2} = \frac{x^2-y^2}{x^2+y^2} \quad ....(5)$$

$\therefore$ From (4)and (5)weget, $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$

4. If $z = e^{ax+by} f(ax - by)$ prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$

Solution:

Given $z = e^{ax+by} f(ax - by)$ ....(1)

Differentiating (1) partially w. r. t. x (treating y as constant) we get,

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax - by)a + f(ax - by)\, e^{ax+by} a$$

$\therefore \frac{\partial z}{\partial x} = ae^{ax+by} f'(ax - by) + az.$ (using (1)). Multiplying both sides by b we get,

$$b\frac{\partial z}{\partial x} = abe^{ax+by} f'(ax - by) + abz \quad ....(2)$$

Differentiating (1) partially w. r. t. y (treating x as constant) we get,

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax - by)(-b) + f(ax - by)\, e^{ax+by} b$$

$\therefore \frac{\partial z}{\partial y} = -be^{ax+by} f'(ax - by) + b\,z$. Multiplying both sides by a we get,

$$a\frac{\partial z}{\partial y} = -abe^{ax+by} f'(ax - by) + ab\,z. \;\ldots\ldots\ldots(3)$$

Adding (2) and (3) we get, $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab\,z.$

5. If $z = f(x + ct) + \emptyset(x - ct)$ prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$.

Solution:

Given $z = f(x + ct) + \emptyset(x - ct) \;\ldots\ldots. (1)$

Differentiating (1) partially w. r. t. x (treating t as constant) we get,

$\frac{\partial z}{\partial x} = f'(x + ct) + \emptyset'(x - ct)$. Again differentiating partially w. r. t. x we get,

$$\frac{\partial^2 z}{\partial x^2} = f''(x + ct) + \emptyset''(x - ct) \;\ldots\ldots\ldots (2)$$

Differentiating (1) partially w. r. t. t (treating x as constant) we get,

$\frac{\partial z}{\partial t} = cf'(x + ct) + \emptyset'(x - ct)(-c)$. Again differentiating partially w. r. t. t we get,

$$\frac{\partial^2 z}{\partial t^2} = c^2 f''(x + ct) + c^2\emptyset''(x - ct) = c^2\{f''(x + ct) + \emptyset''(x - ct)\}$$

$\therefore \quad \frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$ Using (2)

6. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ prove that (i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$ and

(ii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

Solution:

Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$

(i) Differentiating the given equation partially w. r. t. x, y and z we get,

$\frac{\partial u}{\partial x} = \frac{(3x^2-3yz)}{x^3+y^3+z^3-3xyz} \;\ldots\ldots(1)$ $\quad \frac{\partial u}{\partial y} = \frac{(3y^2-3xz)}{x^3+y^3+z^3-3xyz} \;\ldots\ldots(2)$

$\frac{\partial u}{\partial z} = \frac{(3z^2-3xy)}{x^3+y^3+z^3-3xyz} \;\ldots\ldots.(3)$ Adding (1), (2) and (3) we get,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2-yz+y^2-xz+z^2-xy)}{x^3+y^3+z^3-3xyz}=\frac{3(x^2+y^2+z^2-xy-yz-xz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-xz)}=\frac{3}{x+y+z}$$

(ii) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$

$$=\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$=\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}=\frac{-9}{(x+y+z)^2}$$

7. If $x^x y^y z^z=c$, prove that at $x=y=z$, $\frac{\partial^2 z}{\partial x \partial y}=-(x.\log ex)^{-1}$

Solution:

Given $x^x y^y z^z=c$……..(1). Here we have to treat z as function of x and y.

Taking logarithm on both the sides of (1) we get,

$x\log x+y\log y+z\log z=\log c$ ………. (2)

Differentiating (2) partially w. r. t. x we get,

$x\frac{1}{x}+\log x+z\frac{1}{z}\frac{\partial z}{\partial x}+(\log z)\frac{\partial z}{\partial x}=0.$ $\quad\therefore (1+\log x)+\frac{\partial z}{\partial x}(1+\log z)=0$

$\therefore\quad \frac{\partial z}{\partial x}(1+\log z)=-(1+\log x).\quad\therefore\quad \frac{\partial z}{\partial x}=-\frac{(1+\log x)}{(1+\log z)}$ …….. (3)

Similarly by differentiating (2) pa

ially w. r. t. y we get, $\frac{\partial z}{\partial y}=-\frac{(1+\log y)}{(1+\log z)}$ ……..(4)

Differentiating (4) partially w. r. t. x we get,

$$\frac{\partial^2 z}{\partial x\partial y}=-(1+\log y)\left\{\frac{-1}{(1+\log z)^2}\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)\right\}=\frac{(1+\log y)}{z(1+\log z)^2}\left(-\frac{(1+\log x)}{(1+\log z)}\right),\ \text{using (3).}$$

$\therefore\quad \frac{\partial^2 z}{\partial x\partial y}=\frac{-(1+\log x)(1+\log y)}{z(1+\log z)^3}$ When x=y=z we get,

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{(1+\log x)(1+\log x)}{x(1+\log x)^3}=-\frac{1}{x(1+\log x)}=-\frac{1}{x(\log e+\log x)}=-\frac{1}{x(\log ex)}$$

Thus $\frac{\partial^2 z}{\partial x\partial y}=-(x.\log ex)^{-1}$

8. If $x=r\cos\theta$, $y=r\sin\theta$ prove that $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$.

Solution:

Given $x=r\cos\theta$, $y=r\sin\theta$ Squaring and adding we get,

$x^2 + y^2 = r^2 \quad \therefore \quad r^2 = x^2 + y^2$ .... (1)

Differentiating (1) partially w. r. t. x (treating y as constant) we get,

$2r\frac{\partial r}{\partial x} = 2x \quad \therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}$ Again differentiating partially w. r. t. x, we get,

$$\frac{\partial^2 r}{\partial x^2} = \frac{r(1) - x\frac{\partial r}{\partial x}}{r^2} = \frac{r - x\left(\frac{x}{r}\right)}{r^2} = \frac{\frac{r^2 - x^2}{r}}{r^2} = \frac{r^2 - x^2}{r^3} \qquad \therefore \ \frac{\partial^2 r}{\partial x^2} = \frac{r^2 - x^2}{r^3} \ \ldots\ldots\ (2)$$

Differentiating (1) partially w. r. t. y (treating x as constant) we get,

$2r\frac{\partial r}{\partial y} = 2y \quad \therefore \quad \frac{\partial r}{\partial y} = \frac{y}{r}$ Again differentiating partially w. r. t. y, we get,

$$\frac{\partial^2 r}{\partial y^2} = \frac{r(1) - y\frac{\partial r}{\partial y}}{r^2} = \frac{r - y\left(\frac{y}{r}\right)}{r^2} = \frac{\frac{r^2 - y^2}{r}}{r^2} = \frac{r^2 - y^2}{r^3} \qquad \therefore \ \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - y^2}{r^3} \ \ldots\ldots\ (3)$$

Adding (1) and (2) we get,

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3} = \frac{2r^2 - (x^2 + y^2)}{r^3} = \frac{2r^2 - (r^2)}{r^3} = \frac{r^2}{r^3} = \frac{1}{r} \ \text{Using (1)}$$

$$\therefore \ \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} \ \ldots\ldots\ (4)$$

$$\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] = \frac{1}{r}\left\{\frac{x^2}{r^2} + \frac{y^2}{r^2}\right\} = \frac{1}{r}\left(\frac{x^2 + y^2}{r^2}\right) = \frac{1}{r}\left(\frac{r^2}{r^2}\right) = \frac{1}{r}$$

$$\therefore \ \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] = \frac{1}{r} \ \ldots\ldots\ (5)$$

Using (4) and (5) we get, $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$

HOMEWORK:

1. **If $u = x^y$, show that $u_{xy} = u_{yx}$**
2. **If $u = e^{xyz}$ find $\frac{\partial^3 u}{\partial x \partial y \partial z}$**
3. **If $v = (x^2 + y^2 + z^2)^{-1/2}$, prove that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$**
4. **If $u = \log(x^3 + y^3 - x^2 y - xy^2)$, then prove that $u_{xx} + 2u_{xy} + u_{yy} = \frac{-4}{(x+y)^2}$.**
5. **If $u = e^{a\theta}\cos(a \log r)$, prove that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$**
6. **If $u = f(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$ prove that**
   $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r}f'(r)$

TOTAL DERIVATIVES:

Definition:

For a function $u = f(x, y)$ where $x = x(t)$ and $y = y(t)$ that is x and y are functions of an independent variable t, the derivative of u w. r. t. t is called the total derivative of u and is defined by $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

If $u = f(x, y, z)$ where $x = x(t)$, $y = y(t)$ and $z = z(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Problems:

1. If $u = \tan^{-1}\left(\frac{y}{x}\right)$ where $x = e^t - e^{-t}$, $y = e^t + e^{-t}$ find $\frac{du}{dt}$

Solution:

We have $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$ ……(1)

Given $u = \tan^{-1}\left(\frac{y}{x}\right)$…..(2) and $x = e^t - e^{-t}$, $y = e^t + e^{-t}$………(3)

Differentiating (2) partially w. r. t. x and y we get,

$$\frac{\partial u}{\partial x} = \frac{1}{\left(1+\frac{y^2}{x^2}\right)}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{1}{\left(1+\frac{y^2}{x^2}\right)}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

Differentiating (3) w. r. t. t we get,

$\frac{dx}{dt} = e^t + e^{-t} = y$ and $\frac{dy}{dt} = e^t - e^{-t} = x$, using (3). Substituting in (1) we get,

$$\frac{du}{dt} = \frac{-y^2}{x^2+y^2} + \frac{x^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} = \frac{(e^t-e^{-t})^2-(e^t+e^{-t})^2}{(e^t-e^{-t})^2+(e^t+e^{-t})^2}$$

$$\therefore \frac{du}{dt} = \frac{(e^{2t}+e^{-2t}-2e^te^{-t})-(e^{2t}+e^{-2t}+2e^te^{-t})}{(e^{2t}+e^{-2t}-2e^te^{-t})+(e^{2t}+e^{-2t}+2e^te^{-t})} = \frac{-4}{2e^{2t}+2e^{-2t}} = \frac{-2}{e^{2t}+e^{-2t}}$$

2. If $f = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$ find $\frac{df}{dt}$ as a total derivative and verify the result by direct substitution.

Solution:

We have $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$ ………(1)

Given $f = x^2 + y^2 + z^2$……..(2) and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$…..(3)

Differentiating (2) partially w. r. t. x, y and z we get,

$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 2z.$

Differentiating (3) w. r. t. t we get,

$\frac{dx}{dt} = 2e^{2t}, \; \frac{dy}{dt} = -3e^{2t}\sin 3t + 2e^{2t}\cos 3t, \; \frac{dz}{dt} = 3e^{2t}\cos 3t + 2e^{2t}\sin 3t.$

Substituting in (1) we get,

$\frac{df}{dt} = 2x.\,2e^{2t} + 2y(-3e^{2t}\sin 3t + 2e^{2t}\cos 3t\;) + 2z(3e^{2t}\cos 3t + 2e^{2t}\sin 3t)$

$$\therefore \; \frac{df}{dt} = 2e^{2t}.\,2e^{2t} + 2e^{2t}\cos 3t\,(-3e^{2t}\sin 3t + 2e^{2t}\cos 3t\;)$$
$$+2e^{2t}\sin 3t\,(3e^{2t}\cos 3t + 2e^{2t}\sin 3t), \text{ using } (3)$$

$\therefore \; \frac{df}{dt} = 4e^{4t} - 6e^{4t}\cos 3t\sin 3t + 4e^{4t}\cos^2 3t + 6e^{4t}\sin 3t\cos 3t + 4e^{4t}\sin^2 3t$

$\therefore \; \frac{df}{dt} = 8e^{4t}$

Verification by direct substitution:

Substituting (3) in (2) we get,

$f = (e^{2t})^2 + (e^{2t}\cos 3t)^2 + (e^{2t}\sin 3t)^2 = e^{4t} + e^{4t}\cos^2 3t + e^{4t}\sin^2 3t$

$f = e^{4t} + e^{4t}(\cos^2 3t + \sin^2 3t) = e^{4t} + e^{4t} = 2e^{4t}$

Differentiating w. r. t. t we get,

$\frac{df}{dt} = 2(4e^{4t}) = 8e^{4t}. \quad \therefore \; \frac{df}{dt} = 8e^{4t}.$ Hence verified.

3. If $u = e^x \sin(yz)$ where $x = t^2, \; y = t - 1, \; z = \frac{1}{t}$ find $\frac{du}{dt}$ at $t = 1$.

Solution:

We have $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$ ....(1)

Given $u = e^x \sin(yz)$ ......(2) and $x = t^2, \quad y = t - 1, \; z = \frac{1}{t}$ ....(3)

Differentiating (2) partially w. r. t. x, y and z we get,

$\frac{\partial u}{\partial x} = e^x \sin(yz), \quad \frac{\partial u}{\partial y} = e^x \cos(yz)\, z, \quad \frac{\partial u}{\partial z} = e^x \cos(yz) y$

Differentiating (3) w. r. t. t we get, $\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = -\frac{1}{t^2}.$

Substituting in (1) we get,

$\frac{du}{dt} = e^x \sin(yz) . 2t + e^x \cos(yz)\, z(1) + e^x \cos(yz) y \left(-\frac{1}{t^2}\right)$. Using (3) we get,

$\therefore\ \frac{du}{dt} = e^{t^2}\left[2t. \sin\left((t-1)\frac{1}{t}\right) + \cos\left((t-1)\frac{1}{t}\right)\frac{1}{t} + \cos\left((t-1)\frac{1}{t}\right)(t-1)\left(-\frac{1}{t^2}\right)\right]$

Putting $t = 1$ we get, $\frac{du}{dt} = e$

HOMEWORK:

1. **Find $\frac{du}{dt}$ when $u = x^3y^2 + x^2y^3$ with $x = at^2,\ \ y = 2at$**
2. **If $z = xy^2 + x^2y$ when $x = at,\ \ y = 2at$ show that $\frac{dz}{dt} = 18\, a^3t^2$**
3. **Given $u = \sin(x/y)$, $x = e^t$ and $y = t^2$ find $\frac{du}{dt}$ as a function of t.**
   Also verify the result by direct substitution.

5. **Find $\frac{du}{dt}$ for $u = x^2 - y^2$, where $x = e^t \cos t$, $y = e^t \sin t$ at $t = 067.$**
6. **Find $\frac{du}{dt}$ given $u = y^2 - 4ax$, where $x = at^2$, $y = 2at.$**

PARTIAL DERIVATIVES OF COMPOSITE FUNCTIONS:

Chain Rule:

If $f = f(x, y)$ is a function of two variables x and y where x and y are functions of two other variables u and v then

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

If $f = f(x, y, z)$ is a function of three variables x, y, z where x, y, z are functions of three variables u, v, w then

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}, \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v} \quad \text{and}$$

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial w}$$

Problems:

1. **If $z = f(x, y)$, and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$**

   prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Solution:

Given $z = f(x, y)$, where $x = x(u, v)$ and $y = y(u, v)$

We have $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \ldots\ldots (1)$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \ \ldots. (2)$

Given $x = e^u + e^{-v}$, $y = e^{-u} - e^v$ ....(3)

Differentiating (3) partially w. r. t. u and v we get,

$\frac{\partial x}{\partial u} = e^u$, $\frac{\partial x}{\partial v} = -e^{-v}$, $\frac{\partial y}{\partial u} = -e^{-u}$, $\frac{\partial y}{\partial v} = -e^v$. Substituting in (1) and (2) we get,

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u) + \frac{\partial z}{\partial y}(-e^{-u})$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v)$

$\frac{\partial z}{\partial u} = e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} \ldots. (3)$ and $\frac{\partial z}{\partial v} = -e^{-v}\frac{\partial z}{\partial x} - e^v\frac{\partial z}{\partial y}$ ..... (4)

$\therefore$ (3) − (4) gives, $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^u + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^v) = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

2. If $z = f(x, y)$ and $x = e^u \cos v$, $y = e^u \sin v$ prove that

(i) $x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$ (ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$.

Solution:

Given $x = e^u \cos v$, $y = e^u \sin v$ .... (1)

Differentiating (1) partially w. r. t. u and v we get,

$\frac{\partial x}{\partial u} = e^u \cos v$, $\frac{\partial x}{\partial v} = -e^u \sin v$, $\frac{\partial y}{\partial u} = e^u \sin v$, $\frac{\partial y}{\partial v} = e^u \cos v$.

(i) We have $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$

$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u \cos v) + \frac{\partial z}{\partial y}(e^u \sin v) \cdots\cdots\cdots (2)$

And $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^u \sin v) + \frac{\partial z}{\partial y}(e^u \cos v) \cdots\cdots\cdots (3)$

$$\therefore\ x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = (e^u \cos v)\left\{\frac{\partial z}{\partial x}(-e^u \sin v) + \frac{\partial z}{\partial y}(e^u \cos v)\right\}$$

$$+(e^u \sin v)\left\{\frac{\partial z}{\partial x}(e^u \cos v) + \frac{\partial z}{\partial y}(e^u \sin v)\right\}$$

$$= -e^{2u}\sin v \cos v\frac{\partial z}{\partial x} + e^{2u}\cos^2 v\frac{\partial z}{\partial y} + e^{2u}\sin v \cos v\frac{\partial z}{\partial x} + e^{2u}\sin^2 v\frac{\partial z}{\partial y}$$

$$= e^{2u}\frac{\partial z}{\partial y}(\cos^2 v + \sin^2 v) = e^{2u}\frac{\partial z}{\partial y}$$

$$\therefore \quad x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$$

(ii) Squaring and adding (2) and (3) we get,

$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2=\left\{\frac{\partial z}{\partial x}(e^u\cos v)+\frac{\partial z}{\partial y}(e^u\sin v)\right\}^2+\left\{\frac{\partial z}{\partial x}(-e^u\sin v)+\frac{\partial z}{\partial y}(e^u\cos v)\right\}^2$$

$$=\left(\frac{\partial z}{\partial x}\right)^2 e^{2u}\cos^2 v+\left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\sin^2 v+2e^{2u}\frac{\partial z}{\partial x}\cos v\frac{\partial z}{\partial y}\sin v$$

$$+\left(\frac{\partial z}{\partial x}\right)^2 e^{2u}\sin^2 v+\left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\cos^2 v-2e^{2u}\frac{\partial z}{\partial x}\cos v\frac{\partial z}{\partial y}\sin v$$

$$=\left(\frac{\partial z}{\partial x}\right)^2 e^{2u}\cos^2 v+\left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\sin^2 v+\left(\frac{\partial z}{\partial x}\right)^2 e^{2u}\sin^2 v+\left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\cos^2 v$$

$$=e^{2u}\left\{\left(\frac{\partial z}{\partial x}\right)^2\cos^2 v+\left(\frac{\partial z}{\partial y}\right)^2\sin^2 v+\left(\frac{\partial z}{\partial x}\right)^2\sin^2 v+\left(\frac{\partial z}{\partial y}\right)^2\cos^2 v\right\}$$

$$=e^{2u}\left\{\left(\frac{\partial z}{\partial x}\right)^2(\cos^2 v+\sin^2 v)+\left(\frac{\partial z}{\partial y}\right)^2(\sin^2 v+\cos^2 v)\right\}$$

$$\therefore\quad \left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2=e^{2u}\left\{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}$$

$$\therefore\quad \left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]$$

3. If $u=f(x,y)$ and $x=r\cos\theta$, $y=r\sin\theta$ prove that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

Solution:

Given $u=f(x,y)$, where $x=x(r,\theta)$ and $y=y(r,\theta)$

We have $\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$ ….. (1) and $\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$ …..(2}

Given $x=r\cos\theta$, $y=r\sin\theta$ ……. (3)

Differentiating (3) partially w. r. t. r and θ we get,

$\frac{\partial x}{\partial r}=\cos\theta$, $\frac{\partial x}{\partial \theta}=-r\sin\theta$, $\frac{\partial y}{\partial r}=\sin\theta$, $\frac{\partial y}{\partial \theta}=r\cos\theta$.

Substituting in (1) and (2) we get,

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}(\cos\theta)+\frac{\partial u}{\partial y}(\sin\theta)\ \cdots(4)\quad\text{and}\quad \frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}(-r\sin\theta)+\frac{\partial u}{\partial y}(r\cos\theta)$$

$$\therefore\quad \frac{1}{r}\frac{\partial u}{\partial \theta}=-\frac{\partial u}{\partial x}(\sin\theta)+\frac{\partial u}{\partial y}(\cos\theta)\cdots\cdots(5)$$

squaring (4) and (5) adding we get,

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial u}{\partial y}\right)^2 \sin^2\theta + 2\frac{\partial u}{\partial x}\cos\theta\frac{\partial u}{\partial y}\sin\theta$$

$$+\left(\frac{\partial u}{\partial x}\right)^2 \sin^2\theta + \left(\frac{\partial u}{\partial y}\right)^2 \cos^2\theta - 2\frac{\partial u}{\partial x}\sin\theta\frac{\partial u}{\partial y}\cos\theta$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 (\sin^2\theta + \cos^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2 (\sin^2\theta + \cos^2\theta)$$

$$\therefore \quad \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

4. If $u = f(x-y,\ y-z,\ z-x)$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Solution:

Consider $r = x - y,\ s = y - z,\ t = z - x$, then given, $u = f(r, s, t)$

$$\therefore \quad \frac{\partial r}{\partial x} = 1,\ \frac{\partial r}{\partial y} = -1,\ \frac{\partial r}{\partial z} = 0,\ \frac{\partial s}{\partial x} = 0,\ \frac{\partial s}{\partial y} = 1,\ \frac{\partial s}{\partial z} = -1,\ \frac{\partial t}{\partial x} = -1,\ \frac{\partial t}{\partial y} = 0,\ \frac{\partial t}{\partial z} = 1$$

We have $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$, $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-1) = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0) = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \cdots\cdots (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1) = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \cdots\cdots (3)$$

Adding (1), (2), (3) we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = 0.$$

5. If $u = f\left(\frac{y-x}{xy},\ \frac{z-x}{xz}\right)$, prove that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$

Solution:

Consider $r = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$ and $s = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$, then given, $u = f(r,\ s)$

$$\therefore \quad \frac{\partial r}{\partial x} = -\frac{1}{x^2},\ \frac{\partial r}{\partial y} = \frac{1}{y^2},\ \frac{\partial r}{\partial z} = 0,\ \frac{\partial s}{\partial x} = -\frac{1}{x^2},\ \frac{\partial s}{\partial y} = 0,\ \frac{\partial s}{\partial z} = \frac{1}{z^2},$$

We have $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}$, $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y}$, $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z}$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial s}\left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}\frac{\partial u}{\partial r} - \frac{1}{x^2}\frac{\partial u}{\partial s} \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\left(\frac{1}{y^2}\right)+\frac{\partial u}{\partial s}(0)=\frac{1}{y^2}\frac{\partial u}{\partial r}\cdots\cdots(2)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}\left(\frac{1}{z^2}\right)=\frac{1}{z^2}\frac{\partial u}{\partial s}\cdots\cdots(3)$$

$\therefore\ x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=-\frac{\partial u}{\partial r}-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}=0$ Using (1), (2) and (3).

$$\therefore\ x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=0$$

HOMEWORK:

1. **If $u=f(r,\ s,\ t)$ and $r=\frac{x}{y}$, $s=\frac{y}{z}$, $t=\frac{z}{x}$ prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$**
2. **If $u=f(2x-3y,\ 3y-4z,\ 4z-2x)$, prove that $\frac{1}{2}\frac{\partial u}{\partial x}+\frac{1}{3}\frac{\partial u}{\partial y}+\frac{1}{4}\frac{\partial u}{\partial z}=0$.**
3. **If $u=f(e^{y-z},\ e^{z-x},\ e^{x-y})$, prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.**

JACOBIANS:

Definition:

If u and v are functions of two independent variables x and y then the Jacobian of u, v with respect to x, y is denoted by J or $\frac{\partial(u,\ v)}{\partial(x,\ y)}$ or $J\left(\frac{u,\ v}{x,\ y}\right)$ and is

defined by $\frac{\partial(u,\ v)}{\partial(x,\ y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$

similarly, If u, v and w are functions of three independent variables x, y and z then the Jacobian of u, v, w with respect to x, y, z is defined by

$$J\left(\frac{u,v,w}{x,y,z}\right)=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

Problems :

1. If $u=x^2-2y^2$, $v=2x^2-y^2$, where $x=r\cos\theta$ and $y=r\sin\theta$,

then show that $\frac{\partial(u,v)}{\partial(x,y)}=6r^2\sin 2\theta$.

Solution:

Given $u = x^2 - 2y^2$, $v = 2x^2 - y^2$. Differentiating partially w. r. t. x and y

We get, $\frac{\partial u}{\partial x} = 2x$; $\frac{\partial u}{\partial y} = -4y$; $\frac{\partial v}{\partial x} = 4x$; $\frac{\partial v}{\partial y} = -2y$.

We have $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ $\therefore$ $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix} = -4xy + 16xy = 12xy$

Given $x = r\cos\theta$, and $y = r\sin\theta$. $\therefore$ $\frac{\partial(u,v)}{\partial(x,y)} = 12r^2\cos\theta\sin\theta$

$\therefore$ $\frac{\partial(u,v)}{\partial(x,y)} = 6r^2\sin 2\theta$. ($\because \sin 2\theta = 2\sin\theta\cos\theta$)

2. If $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$, then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at (1, -1, 0).

Solution:

Given $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$.

Differentiating partially w. r. t. x, y, z we get, $\frac{\partial u}{\partial x} = 1$; $\frac{\partial u}{\partial y} = 6y$; $\frac{\partial u}{\partial z} = -3z^2$; $\frac{\partial v}{\partial x} = 8xyz$;

$\frac{\partial v}{\partial y} = 4x^2z$; $\frac{\partial v}{\partial z} = 4x^2y$; $\frac{\partial w}{\partial x} = -y$; $\frac{\partial w}{\partial y} = -x$; $\frac{\partial w}{\partial z} = 4z$.

We have $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$ $\therefore$ $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$

$\therefore$ At the point $(1, -1, 0)$, we get, $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 20$

3. If $x = r\sin\theta\cos\emptyset$, $y = r\sin\theta\sin\emptyset$, $z = r\cos\theta$, then show that

$\frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)} = r^2\sin\theta$.

Solution:

Given $x = r\sin\theta\cos\emptyset$, $y = r\sin\theta\sin\emptyset$, $z = r\cos\theta$. Differentiating partially w. r. t. r, θ, ∅ we get, $\frac{\partial x}{\partial r} = \sin\theta\cos\emptyset$; $\frac{\partial x}{\partial \theta} = r\cos\theta\cos\emptyset$; $\frac{\partial x}{\partial \emptyset} = -r\sin\theta\sin\emptyset$; $\frac{\partial y}{\partial r} = \sin\theta\sin\emptyset$;

$\frac{\partial y}{\partial \theta} = r\cos\theta\sin\emptyset$; $\frac{\partial y}{\partial \emptyset} = r\sin\theta\cos\emptyset$; $\frac{\partial z}{\partial r} = \cos\theta$; $\frac{\partial z}{\partial \theta} = -r\sin\theta$; $\frac{\partial z}{\partial \emptyset} = 0$.

We have $\frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \emptyset} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \emptyset} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \emptyset} \end{vmatrix}$

$\therefore \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \sin\theta\cos\emptyset & r\cos\theta\cos\emptyset & -r\sin\theta\sin\emptyset \\ \sin\theta\sin\emptyset & r\cos\theta\sin\emptyset & r\sin\theta\cos\emptyset \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$

$$= \sin\theta\cos\emptyset(0 + r^2\sin^2\theta\cos\emptyset) - r\cos\theta\cos\emptyset(0 - r\sin\theta\cos\theta\cos\emptyset)$$

$$- r\sin\theta\sin\emptyset(- r\sin^2\theta\sin\emptyset - r\cos^2\theta\sin\emptyset).$$

$\therefore \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = r^2\sin^3\theta\cos^2\emptyset + r^2\sin\theta\cos^2\theta\cos^2\emptyset + r^2\sin^3\theta\sin^2\emptyset + r^2\sin\theta\cos^2\theta\sin^2\emptyset.$

$\therefore \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = r^2\sin\theta\cos^2\emptyset(\sin^2\theta + \cos^2\theta) + r^2\sin\theta\sin^2\emptyset(\sin^2\theta + \cos^2\theta).$

$\therefore \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = r^2\sin\theta\cos^2\emptyset + r^2\sin\theta\sin^2\emptyset = r^2\sin\theta(\cos^2\emptyset + \sin^2\emptyset).$

$\therefore \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = r^2\sin\theta.$

4. Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ where $u = x^2 + y^2 + z^2$, $v = xy + yz + zx$, $w = x + y + z$.

Solution:

Given $u = x^2 + y^2 + z^2$, $v = xy + yz + zx$, $w = x + y + z$

Differentiating partially w. r. t. x, y, z we get,

$\frac{\partial u}{\partial x} = 2x;\quad \frac{\partial u}{\partial y} = 2y;\quad \frac{\partial u}{\partial z} = 2z;\quad \frac{\partial v}{\partial x} = y + z;\quad \frac{\partial v}{\partial y} = x + z;\quad \frac{\partial v}{\partial z} = y + x;$

$\frac{\partial w}{\partial x} = 1;\quad \frac{\partial w}{\partial y} = 1;\quad \frac{\partial w}{\partial z} = 1.$

$\therefore \quad J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$\therefore \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$

$$= 2x\{(x + z) - (y + x)\} - 2y\{(y + z) - (y + x)\} + 2z\{(y + z) - (x + z)\}.$$

$$= 2x(z - y) - 2y(z - x) + 2z(y - x) = 2(xz - xy - yz + xy + yz - xz).$$

$\therefore \ \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$

5. If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$, find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3

Solution:

we have Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

Given $y_1 = \frac{x_2 x_3}{x_1}$, $\quad y_2 = \frac{x_3 x_1}{x_2}$, $\quad y_3 = \frac{x_1 x_2}{x_3}$

$\therefore \ \frac{\partial y_1}{\partial x_1} = \frac{-x_2 x_3}{x_1^2} \quad \frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1} \quad \frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1} \quad \frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2} \quad \frac{\partial y_2}{\partial x_2} = \frac{-x_3 x_1}{x_2^2} \quad \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$

$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}. \quad \frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3} \quad \frac{\partial y_3}{\partial x_3} = \frac{-x_1 x_2}{x_3^2}$

$$\therefore \ \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3 x_1}{x_2^2} & \frac{x_1}{x_3} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -1(1-1) - 1(-1-1) + 1(1+1) = 4.$$

Thus $\quad \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = 4.$

HOMEWORK:

1. **If $x = u(1-v)$, $y = uv$, find $\frac{\partial(x,y)}{\partial(u,v)}$**

2. **If $x = a\cosh\xi\cos\eta$, $y = a\sinh\xi\sin\eta$, show that $\frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{a^2}{2}(\cosh 2\xi - \cos 2\eta)$**

3. **If $x = e^u \sec v$, $y = e^u \tan v$, find $\frac{\partial(x,y)}{\partial(u,v)}$.**

Maxima and Minima for functions of two variables:

Definition:

A function $f(x,y)$ is said to have maximum value at $x = a, y = b$, if $f(a,b) > f(a+h, b+k)$ for all small values of h and k. The value $f(a,b)$ is called the maximum value of $f(x,y)$.

A function $f(x,y)$ is said to have minimum value at $x = a, y = b$, if $f(a,b) < f(a+h, b+k)$ for all small values of h and k. The value $f(a,b)$ is called the minimum value of $f(x,y)$.

A maximum or minimum value of a function is called its extreme value.

Conditions for $f(x,y)$ to be maximum or minimum:

The necessary conditions for $f(x,y)$ to have a maximum or minimum value at (a,b) are $f_x(a,b) = 0, f_y(a,b) = 0$

Suppose that $f_x = 0,\ f_y = 0$ and let $\frac{\partial^2 f}{\partial x^2} = f_{xx} = l,\ \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = m,\ \ \frac{\partial^2 f}{\partial y^2} = f_{yy} = n$

then (i) $f(a,b)$ is a maximum value if $ln - m^2 > 0$ and $l < 0$

(ii) $f(a,b)$ is a minimum value if $ln - m^2 > 0$ and $l > 0$

(iii) $f(a,b)$ is not a extreme value if $ln - m^2 < 0$

(iv) If $ln - m^2 = 0$ then $f(x,y)$ fails to have maximum or minimum value and it needs further investigation.

Working Rule to find the maximum or minimum values of $f(x,y)$

1. Find $\frac{\partial f}{\partial x} = f_x$ and $\frac{\partial f}{\partial y} = f_y$ and solve the equations f_x= 0 and f_y=0 to obtain the stationary points.
2. Find $l = \frac{\partial^2 f}{\partial x^2},\ m = \frac{\partial^2 f}{\partial x \partial y}\ ,\ n = \frac{\partial^2 f}{\partial y^2}$ for every stationary point (a,b) obtained in step1.
3. (i) If $ln - m^2 > 0\ and\ l < 0$ at (a,b) then (a,b) is a point of maximum and $f(a,b)$ is maximum value

   (ii) If $ln - m^2 > 0\ and\ l > 0$ at (a,b) then (a,b) is a point of minimum and $f(a,b)$ is minimum value

   (iii) If $ln - m^2 < 0$ at (a,b) then $f(a,b)$ is not an extreme value i.e., there is neither maximum nor a minimum at (a,b). In this case (a,b) is a saddle point.

   (iv) If $ln - m^2 = 0$ at (a,b) no conclusion can be drawn about maximum or minimum and needs further investigation. Similarly examine the other stationary points.

PROBLEMS:

1. Find the extreme values of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

Solution:

$f(x,y) = x^3 + y^3 - 3x - 12y + 20 \quad \dots (1)$

Differentiating (1) partially w. r. t. x and y, we get,

$\frac{\partial f}{\partial x} = f_x = 3x^2 - 3 \quad \dots (2) \qquad \frac{\partial f}{\partial y} = f_y = 3y^2 - 12 \quad \dots (3)$

$\therefore \ \frac{\partial^2 f}{\partial x^2} = 6x = l, \ \frac{\partial^2 f}{\partial x \partial y} = 0 = m, \ \frac{\partial^2 f}{\partial y^2} = 6y = n$

Take $f_x = 0 \ and \ f_y = 0. \quad \therefore \ 3x^2 - 3 = 0 \ and \ 3y^2 - 12 = 0.$

$\therefore \ x^2 = 1 \ and \ y^2 = 4 \quad \therefore \ x = \pm 1$ and $y = \pm 2$

Therefore (1, 2), (1, −2), (−1, 2), (−1, −2) are the stationary points.

	(1, 2)	(1, −2)	(−1, 2)	(−1 , −2)
$l = 6x$	6 > 0	6	−6	−6 < 0
$m = 0$	0	0	0	0
$n = 6y$	12	−12	12	−12
$ln - m^2$	72 > 0	−72	−72	72 > 0
Conclusion	Minimum point	Saddle point	Saddle point	Maximum point

Maximum value of $f(x,y) : f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20 = 38.$

Minimum value of $f(x,y) : f(1,2) = (1)^3 + (2)^3 - 3(1) - 12(2) + 20 = 2.$

Thus the extreme values of the given function are 38 and 2.

2. Examine the function for extreme values $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Solution:

$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2 \quad \dots (1)$

Differentiate partially w. r .t. x and y.

$\frac{\partial f}{\partial x} = f_x = 4x^3 - 4x + 4y \quad \dots (2). \qquad \frac{\partial f}{\partial y} = f_y = 4y^3 + 4x - 4y \quad \dots (3).$

$\therefore \ \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4 = l, \ \frac{\partial^2 f}{\partial x \partial y} = 4 = m, \ \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4 = n.$

Take $f_x = 0$ and $f_y = 0.$

$\therefore \ 4x^3 - 4x + 4y = 0$ and $4y^3 + 4x - 4y = 0.$

$\therefore \ x^3 - x + y = 0 \quad \dots. (4)$ and $y^3 + x - y = 0 \quad \dots.(5)$

Solving (4) and (5) we get $x^3 + y^3 = 0. \quad \therefore \ (x + y)(x^2 - xy + y^2) = 0.$

$x + y = 0. \quad \therefore \ y = -x \ \dots. (6)$ Substituting in (4), we get, $x^3 - x - x = 0.$

$\therefore\ x^3 - 2x = 0. \quad \therefore\ x(x^2-2) = 0. \quad \therefore\ x = 0,\ x^2 = 2. \quad \therefore\ x = \pm\sqrt{2}.$

$\therefore\ x = 0,\ +\sqrt{2},\ -\sqrt{2}$. *Substitute in (5).* $\therefore\ y = 0,\ -\sqrt{2},\ +\sqrt{2}$.

Therefore $(0,\ 0),\ (\sqrt{2},\ -\sqrt{2}),\ (-\sqrt{2},\ \sqrt{2})$ *are the stationary points.*

	(0,0)	$(\sqrt{2},-\sqrt{2})$	$(-\sqrt{2},\sqrt{2})$
$l = 12x^2 - 4$	*−4*	*20 > 0*	*20 > 0*
$m = 4$	*4*	*4*	*4*
$n = 12y^2 - 4$	*−4*	*20*	*20*
$ln - m^2$	*16−16 = 0*	*384 > 0*	*384 > 0*
Conclusion	-	*Minimum point*	*Minimum point*

Since $ln - m^2 > 0,\ l > 0$ *the function is Minimum at both points* $(\sqrt{2},\ -\sqrt{2}),\ (-\sqrt{2},\ \sqrt{2})$.

$\therefore$ *The Minimum value of* $f(x,y)$ *i.e.,* $f(\sqrt{2},\ -\sqrt{2})$ *and* $f(-\sqrt{2},\ \sqrt{2}) = -8$

3. Examine the function $f(x,y) = \sin x + \sin y + \sin(x+y)$ for extreme values.

Solution:

$f(x,y) = \sin x + \sin y + \sin(x+y) \quad \ldots(1)$

Differentiating partially w. r. t. x and y we get,

$\frac{\partial f}{\partial x} = f_x = \cos x + \cos(x+y) \quad , \quad \frac{\partial f}{\partial y} = f_y = \cos y + \cos(x+y)$

$\therefore\ \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y) = l,\ \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y) = m,$

$\frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y) = n.$

Take $f_x = 0$ and $f_y = 0$.

$\therefore\ \cos x + \cos(x+y) = 0 \ \ldots.(2)$ and $\cos y + \cos(x+y) = 0 \ \ldots.(3)$

Subtracting (3) from (2) we get $\cos x - \cos y = 0. \quad \therefore\ \cos x = \cos y. \quad \therefore\ x = y \ldots.(4)$

$\therefore\ (2) \Rightarrow \cos x + \cos 2x = 0. \quad \therefore\ \cos x + 2\cos^2 x - 1 = 0 \Rightarrow 2\cos^2 x + \cos x - 1 = 0.$

$\therefore\ 2\cos^2 x + 2\cos x - \cos x - 1 = 0. \quad \therefore\ 2\cos x(\cos x + 1) - 1(\cos x + 1) = 0.$

$\therefore\ (\cos x + 1)(2\cos x - 1) = 0. \quad \therefore\ 2\cos x - 1 = 0$ and $\cos x + 1 = 0.$

$\therefore\ \cos x = \frac{1}{2} \Rightarrow x = \cos^{-1}\left(\frac{1}{2}\right). \quad \therefore\ x = \frac{\pi}{3}$ and $\cos x = -1 \Rightarrow x = \cos^{-1}(-1). \quad \therefore\ x = \pi.$

Substitute in (4). $\therefore\ y = \frac{\pi}{3}$ and $y = \pi$. $\therefore$ The stationary points are $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and (π, π).

	$\left(\frac{\pi}{3},\frac{\pi}{3}\right)$	(π,π)
$l = -\sin x - \sin(x+y)$	$-\sqrt{3} < 0$	0
$m = -\sin(x+y)$	$\frac{-\sqrt{3}}{2}$	0
$n = -siny - \sin(x+y)$	$-\sqrt{3}$	0
$ln - m^2$	$\frac{9}{4} > 0$	0
Conclusion	Maximum point	-

Since at (π,π), $l = 0$, $m = 0$, $n = 0$, the case needs further investigation.

Hence $f(x,y)$ is Maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and the maximum value is

$f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{3\sqrt{3}}{2}$.

4. Discuss the maxima and minima of $f(x,y) = x^3y^2(1-x-y)$

Solution:

$f(x,y) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$ …. (1)

Differentiating partially w. r. t. x and y we get,

$\frac{\partial f}{\partial x} = f_x = 3x^2y^2 - 4x^3y^2 - 3y^3$ …. (2)

$\frac{\partial f}{\partial y} = f_y = 2x^3y - 2x^4y - 3x^3y^2$ …. (3)

$\therefore \quad \frac{\partial^2 f}{\partial x^2} = l = 6xy^2 - 12xy^3 = 6xy^2(1-2x-y)$.

$\frac{\partial^2 f}{\partial x \partial y} = m = 6x^2y - 8x^3y - 9x^2y^2 = x^2y(6-8x-9y)$.

$\frac{\partial^2 f}{\partial y^2} = n = 2x^3 - 2x^4 - 6x^3y = 2x^3(1-x-3y)$.

Take $f_x = 0$ and $f_y = 0$

$\therefore \quad 3x^2y^2 - 4x^3y^2 - 3y^3 = 0$ and $2x^3y - 2x^4y - 3x^3y^2 = 0$.

$\therefore \quad x^2y^2(3-4x-3y) = 0 => x = 0, y = 0, 3-4x-3y = 0$.

And $x^2y(2-2x-3y) = 0 => x = 0, y = 0, 2-2x-3y = 0$.

Solving $x = 0$ and $3-4x-3y = 0$, $x = 0$ and $2-2x-3y = 0$,

$y = 0$ and $3-4x-3y = 0$, $y = 0$ and $2-2x-3y = 0$,

$2 - 2x - 3y = 0$ and $2 - 2x - 3y = 0$

We get the stationary points $(0,0)$, $(0,1)$, $\left(0, \frac{2}{3}\right)$, $\left(\frac{3}{4}, 0\right)$, $(1, 0)$ *and* $\left(\frac{1}{2}, \frac{1}{3}\right)$

Now $ln - m^2 = [6xy^2(1 - 2x - y)][2x^3(1 - x - 3y)] - [x^2y(6 - 8x - 9y)]^2$

$\therefore \quad ln - m^2 = (x^2y)^2[12(1 - 2x - y)(1 - x - 3y) - (6 - 8x - 9y)^2]$

At all points except $\left(\frac{1}{2}, \frac{1}{3}\right)$, we get $ln - m^2 = 0$ i.e., there are no extreme value.

$\therefore \quad At \left(\frac{1}{2}, \frac{1}{3}\right), \ ln - m^2 = \left(\frac{1}{4}.\frac{1}{3}\right)^2 \left\{12\left[1 - 2\left(\frac{1}{2}\right) - \frac{1}{3}\right]\left[1 - \frac{1}{2} - 3\left(\frac{1}{3}\right) - \left[6 - 8\left(\frac{1}{2}\right) - 9\left(\frac{1}{3}\right)\right]^2\right]\right\}$

$\therefore \quad ln - m^2 = \frac{1}{144}\{2 - 1\} = \frac{1}{144} > 0$ and $l = \frac{1}{3}\left(1 - 1 - \frac{1}{3}\right) = -\frac{1}{9} < 0.$

$\therefore \quad f(x, y)$ is Maximum at $\left(\frac{1}{2}, \frac{1}{3}\right)$.

Thus the maximum value of $f(x, y) = f\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{8}.\frac{1}{9}\right)\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{72}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$.

5. A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimension of the box requiring least material for its construction.

Solution:

Let x ft, y ft, z ft be the dimension of the box and let S be the surface of the box then we have

$S = xy + 2yz + 2zx \ \ldots (1)$ (Since open at the top)

Given that the volume $xyz = 32$. $\quad \therefore \quad z = \frac{32}{xy} \ldots (2)$ Substituting (2) in (1), we get,

$S = xy + \frac{64}{x} + \frac{64}{y} = f(x, y)$. Differentiating partially w. r. t. x and y, we get,

$\frac{\partial f}{\partial x} = f_x = y - \frac{64}{x^2}, \quad \frac{\partial f}{\partial y} = f_y = x - \frac{64}{y^2}.$

$\therefore \quad l = \frac{\partial^2 f}{\partial x^2} = \frac{128}{x^3}, \ m = \frac{\partial^2 f}{\partial x \partial y} = 1, \ n = \frac{\partial^2 f}{\partial y^2} = \frac{128}{y^3}.$

Take $f_x = 0$ *and* $f_y = 0$. $\quad \therefore \quad y - \frac{64}{x^2} = 0$ *and* $x - \frac{64}{y^2} = 0.$

$\therefore \quad y = \frac{64}{x^2} \ldots (3)$ and $x = \frac{64}{y^2} \ldots (4)$.

Substituting (3) in (4), we get, $x = \frac{64}{4096/x^4} \quad \therefore \quad x - \frac{x^4}{64} = 0. \quad \therefore \quad x\left(1 - \frac{x^3}{64}\right) = 0.$

$\therefore \quad x^3 = 64. \quad \therefore \quad x = 4.$ Put in (3). $\quad \therefore \quad y = \frac{64}{16}. \quad \therefore \quad y = 4.$

$\therefore$ The stationary point is $(4, 4)$.

$\therefore$ At $(4,\ 4)$, $l = \frac{128}{64} = 2 > 0$, $ln - m^2 = 4 - 1 = 3 > 0$.

Thus $f(x, y)$ is minimum i.e., S is minimum when x=4, y=4.

From (2), $z = \frac{32}{xy} = \frac{32}{16} = 2$.

$\therefore$ The dimensions of the box for least material for its construction are $(4, 4, 2)$.

HOME WORK:

1. Find the maximum and minimum values of the function $\boldsymbol{f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x}$**.**

2. Examine the function $\boldsymbol{f(x, y) = xy(a - x - y)}$ **for extreme values.**

3. Examine the function $\boldsymbol{f(x, y) = 1 + \sin(x^2 + y^2)}$ **for extremum.**

4. Show that the function $\boldsymbol{f(x, y) = x^3 + y^3 - 63(x + y) + 12xy}$ **is maximum (−7, −7) and minimum (3, 3).**

5. Find the maximum and minimum values of $\boldsymbol{f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}}$ **.**