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NAGARJUNA
COLLEGE OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

ADVANCED CALCULUS AND NUMERICAL METHODS

(COURSE CODE 22MATS21/22MATE21/22MATC21)

CLASS NOTES FOR II SEM B.E.

MODULE-5

PARTIAL DIFFERENTIAL EQUATIONS

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PARTIAL DIFFERENTIAL EQUATIONS

SYLLABUS:
Formation of PDE's by elimination of arbitrary constants and functions. Solution of non-homogeneous PDE by direct integration. Homogeneous PDEs involving derivative with respect to one independent variable only. Solution of PDE by method of separation of variables. Solution of one-dimensional heat equation by the method of separation of variables.

Introduction:

Many Engineering problems like vibration of string, heat conduction electrostatics, involve two or more variables. Analysis of these problems leads to partial derivatives and equation involving them.

In this module, we discuss the formation of PDE analogous to that of formation of ODE and then discuss some methods to solve PDE. Also we discuss one dimensional wave and heat equation.

Definition:

An equation involving one are more partial derivatives of a function of two or more variables is called a PDE.

Formation of PDE by eliminating the arbitrary constants:

Given a relation of the form $f(x, y, z, a, b) = 0$…… (1), where z is a function of x, y and a, b, are arbitrary constants.

Differentiate (1) partially w. r. t. x and w. r. t. y. By using these results eliminate the arbitrary constants a *and* b from (1). The resulting relation is the PDE of (1).

The following standard notations are used for the partial derivatives of z w. r. t. two independent variables x and y.

$$p = \frac{\partial z}{\partial x} = z_x , \quad q = \frac{\partial z}{\partial y} = z_y, \quad r = \frac{\partial^2 z}{\partial x^2} = z_{xx} , \quad s = \frac{\partial^2 z}{\partial x\, \partial y} = z_{xy} , \quad t = \frac{\partial^2 z}{\partial y^2} = z_{yy} .$$

NOTE:

If the elimination of number of arbitrary constants exceeds the number of independent variables, then second or higher order PDE's will be formed.

Problems:

1. Form PDE by eliminating arbitrary constants a and b from $z = (x + a)(y + b)$.

Solution:

Given $z = (x + a)(y + b)$…… (1) Differentiate partially w.r.t x and w.r.t y.

$\therefore \; p = \frac{\partial z}{\partial x} = (y + b)$…….. (2) and $q = \frac{\partial z}{\partial y} = (x + b)$….. (3)

Using (2) & (3) in (1), we obtain $z = pq$

Thus, $z = pq$ is the required PDE.

2. Form PDE by eliminating arbitrary constants from $z = (x^2 + a)(y^2 + b)$.

Solution:

Given $z = (x^2 + a)(y^2 + b) \ldots\ldots\ldots (1)$ Differentiate partially w. r. t. x and y .

$\therefore \frac{\partial z}{\partial x} = 2x(y^2 + b) \ldots\ldots (2)$ *and* $\frac{\partial z}{\partial y} = 2y(x^2 + a) \ldots\ldots\ldots (3)$. *Multiply* (2) *and* (3).

$\therefore \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right) = 4xy(x^2 + a)(y^2 + b)$. *Using* (1), *we get,*

$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right) = 4xyz$. This is the required PDE.

3. Form PDE by eliminating arbitrary constants from $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

Solution:

Given $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \ldots\ldots\ldots (1)$. Differentiate (1) partially w.r.t x and w.r.t y.

$\therefore 2\frac{\partial z}{\partial x} = 2p = \frac{2x}{a^2}$ or $a^2 = \frac{x}{p} \ldots\ldots\ldots (2)$.

And $2\frac{\partial z}{\partial y} = 2q = \frac{2y}{b^2}$ or $b^2 = \frac{y}{q} \ldots\ldots\ldots (3)$.

Using (2) & (3) in (1), we obtain, $2z = x^2\frac{p}{x} + y^2\frac{q}{y}$

Thus, $2z = px + qy$ is the required PDE.

4. Form PDE by eliminating arbitrary constants from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solution:

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots (1)$ Differentiate (1) partially w.r.t x and w.r.t y.

$\therefore \frac{2x}{a^2} + \frac{2z}{c^2}p = 0$ and $\frac{2y}{b^2} + \frac{2z}{c^2}q = 0$.

$\therefore \frac{x}{a^2} + \frac{zp}{c^2} = 0 \ldots (2)$ and $\frac{y}{b^2} + \frac{zq}{c^2} = 0 \ldots\ldots (3)$

Again differentiate (2) partially w.r.t x.

$\therefore \frac{1}{a^2} + \frac{1}{c^2}(zr + p^2) = 0 \ldots\ldots (4)$ $\quad \because \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = r.$

Now, from (2), $\frac{x}{a^2} = \frac{-zp}{c^2}$ or $\frac{1}{a^2} = \frac{-zp}{c^2 x}$. Substituting in (4), we get,

$\frac{-zp}{c^2 x} = \frac{-1}{c^2}(zr + p^2)$ or $zp = x(zr + p^2)$

Thus, $z\frac{\partial z}{\partial x} = xz\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2$ is the required PDE.

5. Form PDE by eliminating arbitrary constants from $ax^2 + by^2 + z^2 = 1$.

Solution:

Given $ax^2 + by^2 + z^2 = 1 \dots \dots (1)$

Differentiate the above equation partially w.r.to x and y and put $\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q$

$\therefore\ 2ax + 2z\frac{\partial z}{\partial x} = 0. \quad \therefore\ a = -\frac{zp}{x} \dots ..(2)$ and $2ay + 2z\frac{\partial z}{\partial y} = 0. \quad \therefore\ b = -\frac{zq}{y} \dots ..(3)$

Substitute (2) and (3) in (1). $\quad \therefore\ -xpz - zqy + z^2 = 1. \quad \therefore\ z^2 = 1 + pxz + zqy.$

$\therefore\ z^2 - 1 = z(xp - yq)$ is the required PDE.

6. Form PDE by eliminating arbitrary constants from $z = xy + y\sqrt{x^2 - a^2} + b.$

Solution:

Given $z = xy + y\sqrt{x^2 - a^2} + b \dots \dots \dots (1).$

Differentiate the above equation partially w.r.to x and y and put $\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q.$

$\frac{\partial z}{\partial x} = y + \frac{2xy}{2\sqrt{x^2-a^2}}. \quad \therefore\ p = \frac{\partial z}{\partial x} = y + \frac{xy}{\sqrt{x^2-a^2}} \dots \dots ..(2).$

And $q = \frac{\partial z}{\partial y} = x + \sqrt{x^2 - a^2} \dots \dots ..(3).$ From (3), $\quad q - x = \sqrt{x^2 - a^2} \dots ..(4).$

Using (4) in (2), we get, $\quad p = y + \frac{xy}{q-x}. \quad \therefore\ p - y = \frac{xy}{q-x}.$

$\therefore\ (p - y)(q - x) = xy. \quad \therefore\ pq - px - qy + xy = xy.$

$\therefore\ pq = px + qy$ is the required PDE.

7. Form PDE by eliminating arbitrary constants from $z = a\log(x^2 + y^2) + b.$

Solution:

Given $z = a\log(x^2 + y^2) + b \dots \dots \dots (1)$

Differentiate the above equation partially w.r.to x and y and put $\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q.$

$\therefore\ \frac{\partial z}{\partial x} = \frac{2ax}{x^2+y^2}. \quad \therefore\ a = \frac{p(x^2+y^2)}{2x} \dots \dots \dots .(2)$ and $\frac{\partial z}{\partial y} = \frac{2ay}{x^2+y^2}.$

$\therefore\ \quad a = \frac{q(x^2+y^2)}{2y} \dots \dots \dots .(3)$

$Equating\ eq(2) and\ eq(3),\ we\ get,\ \frac{p}{x} = \frac{q}{y}.$

$\therefore\ py - qx = 0\ is\ the\ required\ PDE.$

8. Form PDE by eliminating arbitrary constants from $2z = (x + a)^{\frac{1}{2}} + (y - a)^{\frac{1}{2}} + b.$

Solution:

Given $2z = (x-a)^{\frac{1}{2}} + (y-a)^{\frac{1}{2}} + b \ldots\ldots\ldots\ldots\ldots\ldots (1)$

Differentiate the above equation partially w.r.to x and y and put $\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q.$

$2\frac{\partial z}{\partial x} = \frac{1}{2}(x+a)^{-\frac{1}{2}}. \quad \therefore \quad (x+a)^{\frac{1}{2}} = \frac{1}{4p}. \quad \therefore \quad (x+a) = \frac{1}{16p^2} \ldots\ldots.(2).$

And $2\frac{\partial z}{\partial y} = \frac{1}{2}(y-a)^{-\frac{1}{2}}. \quad \therefore \quad (y-a)^{\frac{1}{2}} = \frac{1}{4q}.$

$\therefore \quad (y-a) = \frac{1}{16q^2} \ldots\ldots.(3). Adding\ eq(2)\ and\ eq(3),\ we\ get,$

$x + y = \frac{1}{16}\left(\frac{1}{p^2} + \frac{1}{q^2}\right)$ is the required PDE.

9. Form PDE by eliminating arbitrary constants from $z = (x-a)^2 + (y-b)^2$.

Solution:

Given $z = (x-a)^2 + (y-b)^2 \ldots\ldots\ldots (1).$

Differentiate the above equation partially w.r.to x and y and put $\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q.$

$\therefore \quad \frac{\partial z}{\partial x} = 2(x-a). \quad \therefore \quad x - a = \frac{p}{2} \ldots\ldots\ldots.(2).$

And $\frac{\partial z}{\partial y} = 2(y-b). \quad \therefore \quad y - b = \frac{q}{2} \ldots..(3)$

Substituting (2)*and* (3) *in* (1), *we get*, $z = \frac{p^2}{4} + \frac{q^2}{4}$ *is the required PDE.*

10. Find the PDE of the family of all spheres whose centre lies on the plane $z = 0$ *and have* a constant radius 'r'.

Solution:

The coordinates of the centre of the sphere can be taken as (a, b, 0) where a and b are arbitrary, and r is the constant radius.

$\therefore$ The equation of the sphere is given by $(x-a)^2 + (y-b)^2 + (z-0)^2 = r^2$

$\therefore \quad (x-a)^2 + (y-b)^2 + z^2 = r^2$.......(1)

Where a and b are arbitrary constants has to be eliminated.

Differentiating (1) partially w.r.t x and y, we get, $2(x-a) + 2zp = 0$ and $2(y-b) + 2zq = 0$.

Dividing these two equations by 2, $(x-a) = -zp$ *and* $(y-b) = -zq$.

Substituting these values in (1), we get, $z^2(p^2 + q^2 + 1) = r^2$ is the required PDE.

11. Find the partial differential equation representing all the planes that are a constant

perpendicular distance a from the origin.

Solution:

The equation of the plane in normal form is given by $lx + my + nz = a \ldots..(1)$

Where $l^2 + m^2 + n^2 = 1$. $\therefore$ $n = \sqrt{1 - l^2 - m^2}$.

$\therefore$ (1) becomes, $lx + my + \sqrt{1 - l^2 - m^2}\ z = a$ …….. (2).

Differentiating partially w.r.to x , we get, $l + \sqrt{1 - l^2 - m^2}\ p = 0$….(3).

Differentiating partially w.r.to y , we get, $m + \sqrt{1 - l^2 - m^2}\ q = 0$….(4).

Now we have to eliminate l, m from (2) ,(3) and (4).

From (3) and (4), we have $l = -\sqrt{1 - l^2 - m^2}$ p and $m = -\sqrt{1 - l^2 - m^2}\ q$.

Squaring and adding above equations, $l^2 + m^2 = (1 - l^2 - m^2)(p^2 + q^2)$.

$\therefore$ $l^2 + m^2 = (p^2 + q^2) - (l^2 + m^2)(p^2 + q^2)$

$\therefore$ $(l^2 + m^2) + (l^2 + m^2)(p^2 + q^2) = (p^2 + q^2)$

$\therefore$ $(l^2 + m^2)(1 + p^2 + q^2) = p^2 + q^2$. $\therefore$ $l^2 + m^2 = \frac{p^2+q^2}{1+p^2+q^2}$

$\therefore$ $1 - l^2 - m^2 = 1 - \frac{p^2+q^2}{1+p^2+q^2} = \frac{1}{1+p^2+q^2}$.

Also $l = -\frac{p}{\sqrt{1+p^2+q^2}}$ and $m = -\frac{q}{\sqrt{1+p^2+q^2}}$

Substituting the values of l, m and $1 - l^2 - m^2$ in (2), we get,

$$-\frac{px}{\sqrt{1 + p^2 + q^2}} - \frac{qy}{\sqrt{1 + p^2 + q^2}} + \frac{z}{\sqrt{1 + p^2 + q^2}} = a$$

Or $z = px + qy + a\sqrt{1 + p^2 + q^2}$ is the required PDE.

12. Form a partial differential equation by eliminating the arbitrary constants a, b, c from the relation $z = ax + by + cxy$.

Solution:

Given $z = ax + by + cxy$……(1)

Differentiate (1) partially w.r.to x and y and put $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$.

$\therefore$ $p = \frac{\partial z}{\partial x} = a + cy$…….(2) and $q = \frac{\partial z}{\partial y} = b + cx$…….(3).

Differentiate (2) partially w.r.to y. $\therefore$ $s = \frac{\partial^2 z}{\partial x \partial y} = c$……(4).

Substitute (4) in (2) and (3).

$\therefore \quad a = p - ys$ and $b = q - xs$. Substitute in (1).

$\therefore \quad z = (p - ys)x + (q - xs)y + xys. \qquad \therefore \quad z = px + qy - xys$ *is the required PDE.*

HOME WORK:

1. Form a partial differential equation by eliminating the arbitrary constants a and b from the following relations:

(i) $z = a\log\left[\frac{b(y-1)}{(1-x)}\right]$ **(ii)** $(x-a)^2 + (y-b)^2 = z^2\cot^2\alpha$, α **is a fixed constant.**

2. Form a PDE of all planes having equal x and y intercepts.

3. Find the PDE of all spheres whose centres lie on the z-axis'

Formation of PDE by eliminating the arbitrary functions:

Given a relation of the form $z = f(x, y)$ … (1), where z is a function of x, y.

Differentiate (1) partially w. r. t. x and w. r. t. y. By using these results eliminate the arbitrary function f from (1). The resulting relation is the PDE of (1).

If there are two arbitrary functions to eliminate, then we will get second order PDE.

Problems:

1. Form the PDE by eliminating the arbitrary function from $z = (x+y)\emptyset(x^2 - y^2)$.

Solution:

Given $z = (x+y)\emptyset(x^2 - y^2) \ldots\ldots (1)$. Differentiate (1) partially w.r.to x and y.

$\therefore \quad \frac{\partial z}{\partial x} = p = 2x(x+y)\emptyset'(x^2 - y^2) + \emptyset(x^2 - y^2)$……….(2).

And $\frac{\partial z}{\partial y} = q = -2y(x+y)\emptyset'(x^2 - y^2) + \emptyset(x^2 - y^2)$…………(3)

Multiply (2) by y and (3) by x.

$\therefore \quad py = 2xy(x+y)\emptyset'(x^2 - y^2) + y\emptyset(x^2 - y^2)$…….(4)

and $qx = -2xy(x+y)\emptyset'(x^2 - y^2) + x\emptyset(x^2 - y^2)$……….(5)

Adding (4) and (5), we get, $py + qx = (x+y)\emptyset(x^2 - y^2)$. *Using* (1), *we get,*

$py + qx = z$ *is the required PDE.*

2. Form the PDE by eliminating the arbitrary function from $z = e^y f(x+y)$.

Solution:

Given $z = e^y f(x+y) \ldots\ldots (1)$ Differentiate (1) partially w.r.to x and y.

$\therefore \; \frac{\partial z}{\partial x} = p = e^y f'(x+y)$..........(2) and $\frac{\partial z}{\partial y} = q = e^y f'(x+y) + e^y f(x+y)$..........(3)

Substituting (2) and (1) in (3), we get, $q = p + z$.

$\therefore \; p = q - z \quad is\ the\ required\ PDE.$

3. Form the PDE by eliminating the arbitrary function from $z = f(x^2 - y^2)$.

Solution:

Given $z = f(x^2 - y^2) \ldots \ldots (1)$. Differentiate (1) partially w.r.to x and y.

$\therefore \; \frac{\partial z}{\partial x} = p = 2xf'(x^2 - y^2)$..........(2) and $\frac{\partial z}{\partial y} = q = -2yf'(x^2 - y^2)$............(3)

$Dividing\ (2)\ by\ (3)$, we get, $\frac{p}{q} = -\frac{x}{y}$. $\therefore \; py + qx = 0\ is\ the\ required\ PDE.$

4. Form the PDE by eliminating the arbitrary function from $z = f\left(\frac{xy}{z}\right)$.

Solution:

Given $z = f\left(\frac{xy}{z}\right)$......(1). Differentiate (1) partially w.r.to x and y.

$\therefore \; \frac{\partial z}{\partial x} = p = f'\left(\frac{xy}{z}\right)\left\{\frac{y}{z} - \frac{xy}{z^2}\frac{\partial z}{\partial x}\right\}$............(2) and $\frac{\partial z}{\partial y} = q = f'\left(\frac{xy}{z}\right)\left\{\frac{x}{z} - \frac{xy}{z^2}\frac{\partial z}{\partial y}\right\}$ ....(3)

Dividing (2) by (3), we get, $\frac{p}{q} = \frac{\left\{\frac{y}{z} - \frac{xy}{z^2}p\right\}}{\left\{\frac{x}{z} - \frac{xy}{z^2}q\right\}}$. $\therefore \; \frac{px}{z} - \frac{pqxy}{z^2} = \frac{qy}{z} - \frac{pqxy}{z^2}$.

$\therefore \; px = qy$ is the required PDE.

5. Form the PDE by eliminating the arbitrary function from $xyz = f(x+y+z)$

Solution:

Given $xyz = f(x+y+z)$....... (1). Differentiate (1) partially w.r.to x and y.

$\therefore \; xy\frac{\partial z}{\partial x} + yz = f'(x+y+z)\left(1 + \frac{\partial z}{\partial x}\right)$. $\therefore \; pxy + yz = f'(x+y+z)(1+p)$.........(2)

$\therefore \; xy\frac{\partial z}{\partial y} + xz = f'(x+y+z)\left(1 + \frac{\partial z}{\partial y}\right)$. $\therefore \; qxy + xz = f'(x+y+z)(1+q) \ldots \ldots (3)$

Divide (2) by (3). $\therefore \; \frac{pxy+yz}{qxy+xz} = \frac{1+p}{1+q}$.

$\therefore \; pxy + yz + pqxy + qyz = qxy + xz + pqxy + pxz.$

$\therefore \; pxy - pxz + qyz - qxy = z(x-y).$

$\therefore \; x(y-z)p + y(z-x)q = z(x-y)$ is the required PDE.

6. Form the PDE by eliminating the arbitrary function from $x + y + z = f(x^2 + y^2 + z^2)$.

Solution:

Given $x + y + z = f(x^2 + y^2 + z^2) \dots\dots(1)$ Differentiate (1) partially w.r.to x and y.

$\therefore\ 1 + \frac{\partial z}{\partial x} = f'(x^2 + y^2 + z^2)\left(2x + 2z\frac{\partial z}{\partial x}\right).$

$\therefore\ 1 + p = f'(x^2 + y^2 + z^2)(2x + 2zp) \dots\dots(2).$

And $1 + \frac{\partial z}{\partial y} = f'(x^2 + y^2 + z^2)\left(2y + 2z\frac{\partial z}{\partial y}\right).$

$\therefore\ 1 + q = f'(x^2 + y^2 + z^2)(2y + 2zq)$......(3)

Dividing (2) by (3), we get, $\frac{1+p}{1+q} = \frac{2x+2zp}{2y+2zq}$. $\therefore\ \frac{1+p}{1+q} = \frac{x+zp}{y+zq}$.

$\therefore\ y + zq + py + pqz = x + xq + zp + pqz$

$\therefore\ p(y - z) + q(z - x) = x - y$ is the required PDE.

7. Form the PDE by eliminating the arbitrary function from $z = f(sinx + cosy)$

Solution:

Given $z = f(sinx + cosy) \dots\dots(1)$. Differentiate (1) partially w.r.to x and y.

$\therefore\ \frac{\partial z}{\partial x} = p = cosx\, f'(sinx + cosy)$..........(2). $\therefore\ \frac{\partial z}{\partial y} = q = -sinyf'(sinx + cosy)$..........(3)

Dividing (2) *by* (3), *we get*, $\therefore\ \frac{p}{q} = -\frac{cosx}{siny}$.

$\therefore\ psiny + qcosx = 0$ *is the required PDE*.

8. Form the PDE by eliminating the arbitrary function from $z = f(x^2 + y^2)$.

Solution:

Given $z = f(x^2 + y^2) \dots\dots(1)$. Differentiate (1) partially w.r.to x and y.

$\therefore\ \frac{\partial z}{\partial x} = p = 2xf'(x^2 + y^2)$..........(2) and $\frac{\partial z}{\partial y} = q = 2yf'(x^2 + y^2)$..........(3)

Dividing (2) *by* (3), we get, $\frac{p}{q} = \frac{x}{y}$. $\therefore\ py - qx = 0$ *is the required PDE*.

9. Form the PDE by eliminating the arbitrary function from $z = e^{my}\emptyset(x - y)$.

Solution:

Given $z = e^{my}\emptyset(x - y) \dots\dots(1)$. Differentiate (1) partially w.r.to x and y.

$\therefore\ \frac{\partial z}{\partial x} = p = e^{my}\emptyset'(x - y)$..........(2) and

$\frac{\partial z}{\partial y} = q = -e^{my}\emptyset'(x - y) + me^{my}\emptyset(x - y)$.......(3)

Substitute (2) and (1) in (3). $\therefore\ q = -p + mz$. $\therefore\ p + q = mz$ is the required PDE.

10. Form the PDE by eliminating the arbitrary function from
$lx + my + nz = \emptyset(x^2 + y^2 + z^2)$.

Solution:

Given $lx + my + nz = \emptyset(x^2 + y^2 + z^2) \ldots \ldots .(1)$ Differentiate (1) partially w.r.to x and y.

$\therefore \quad l + n\frac{\partial z}{\partial x} = \emptyset'(x^2 + y^2 + z^2)\left(2x + 2z\frac{\partial z}{\partial x}\right).$

$\therefore \quad l + np = \emptyset'(x^2 + y^2 + z^2)(2x + 2zp)$.........(2)

And $m + n\frac{\partial z}{\partial y} = \emptyset'(x^2 + y^2 + z^2)\left(2y + 2z\frac{\partial z}{\partial y}\right).$

$\therefore \; m + nq = \emptyset'(x^2 + y^2 + z^2)(2y + 2zq)$...(3). Dividing (2) by (3), we get,

$\frac{l+np}{m+nq} = \frac{2x+2zp}{2y+2zq}$. $\therefore \; \frac{l+np}{m+nq} = \frac{x+zp}{y+zq}$.

$\therefore \quad ly + lzq + npy + nzpq = mx + mpz + nqx + nzpq.$

$\therefore \quad ly + lzq + npy - mpz - nqx = mx.$

$\therefore \quad p(ny - mz) + q(lz - nx) = mx - ly \textit{ is the required PDE}.$

11. Form the PDE by eliminating the arbitrary function from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$.

Solution:

Given $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$..............(1). Differentiate (1) partially w.r.to x and y.

$\therefore \; \frac{\partial z}{\partial x} = p = -\frac{2}{x^2}f'\left(\frac{1}{x} + \log y\right)$..............(2). And

$\frac{\partial z}{\partial y} = q = 2y + \frac{2}{y}f'\left(\frac{1}{x} + \log y\right)$. Multiply by y. $\therefore \; qy = 2y^2 + 2f'\left(\frac{1}{x} + \log y\right)$............(3)

From (2) we have, $f'\left(\frac{1}{x} + \log y\right) = -\frac{px^2}{2}$ ....(4.) Substitute (4) in (3), we get,

$\therefore \; qy = 2y^2 - px^2$. $\therefore \; px^2 + qy = 2y^2$ is the required PDE.

12. Form the PDE by eliminating the arbitrary function $z = e^{ax+by}f(ax - by)$.

Solution:

Given $z = e^{ax+by}f(ax - by) \ldots \ldots (1)$. Differentiate (1) partially w.r.to x and y.

$\therefore \; \frac{\partial z}{\partial x} = p = ae^{ax+by}f(ax - by) + a\,be^{ax+by}f'(ax - by).$

$\therefore \; p = ae^{ax+by}\,[f(ax - by) + f'(ax - by)]$...........(2)

$\therefore \; \frac{\partial z}{\partial y} = q = be^{ax+by}f(ax - by) - be^{ax+by}f'(ax - by).$

$\therefore \quad q = be^{ax+by}\left[f(ax-by) - f'(ax-by)\right]$...........(3)

Multiply (2) by b and multiply (3) by a.

$\therefore \; pb = abe^{ax+by}\left[f(ax-by) + f'(ax-by)\right]$...........(4)

$\therefore \; aq = abe^{ax+by}\left[f(ax-by) - f'(ax-by)\right]$...........(5).

$Add\ (4)\ and\ (5). \quad \therefore \quad pb + aq = 2abe^{ax+by}f(ax-by).$

$\therefore \; pb + aq = 2abz$ is the required PDE.

13. Form the PDE by eliminating the arbitrary functions from $z = xf(y) + yg(x)$

Solution:

Given $z = xf(y) + yg(x) \ldots.(1)$. Differentiate (1) partially w.r.to x and y.

$\therefore \; \frac{\partial z}{\partial x} = p = f(y) + yg'(x)$..........(2). And $\frac{\partial z}{\partial y} = q = xf'(y) + g(x)$.............(3).

$\therefore \; \frac{\partial^2 z}{\partial x\,\partial y} = s = f'(y) + g'(x)$..........(4)

Multiplying (2) by x and (3) by y and adding we get,

$xp + yq = xf(y) + xyg'(x) + xyf'(y) + yg(x)$

$\therefore \; xp + yq = [\,xf(y) + yg(x)] + xy[f'(y) + g'(x)]$

$\therefore \; xp + yq = z + xys.$ Using (1) and (4).

This is the required PDE.

14. Form the PDE by eliminating the arbitrary functions from $z = f(x+at) + g(x-at)$

Solution:

Given $z = f(x+at) + g(x-at)\ldots\ldots.(1)$. Differentiate (1) partially w.r.t x and w.r.t t.

$\therefore \; \frac{\partial z}{\partial x} = f'(x+at) + g'(x-at), \quad \frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at) \ldots\ldots.(2).$

And $\frac{\partial z}{\partial t} = af'(x+at) - ag'(x-at),$

$\frac{\partial^2 z}{\partial t^2} = a^2 f''(x+at) + a^2 g''(x-at) = a^2\frac{\partial^2 z}{\partial x^2}.$ [By using (2)]

Thus the desired PDE is $\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}.$

15. Form the PDE by eliminating the arbitrary functions from $z = f(y+2x) + g(y-3x).$

Solution:

Given $z = f(y+2x) + g(y-3x)$.........(1). Differentiate (1) partially w.r.to x and y.

$\therefore \quad \frac{\partial z}{\partial x} = p = 2f'(y+2x) - 3g'(y-3x)$......(2).

And $\quad \frac{\partial z}{\partial y} = q = f'(y+2x) + g'(y-3x)$.........(3).

Differentiate (2) and (3) partially w.r.to x and y.

$\therefore \quad \frac{\partial^2 z}{\partial x^2} = 4f''(y+2x) + 9g''(y-3x)$......(4) and

$\frac{\partial^2 z}{\partial y^2} = f''(y+2x) + g''(y-3x)$.......(5).

Differentiate (2) partially w.r.to y. $\quad \therefore \quad \frac{\partial^2 z}{\partial x \partial y} = 2f''(y+2x) - 3g''(y-3x)$......(6).

Adding (4) and (6), we get, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = 6f''(y+2x) + 6g''(y-3x)$.

$Using\ eq(5),\ we\ get,\ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = 6\frac{\partial^2 z}{\partial y^2}$ is the required PDE.

16. Form the PDE by eliminating the arbitrary functions from
$\emptyset(x+y+z,\ x^2+y^2-z^2) = 0.$

Solution:

Given $\emptyset(x+y+z, x^2+y^2-z^2) = 0$.......(1).

Let $u = x+y+z$ and $v = x^2+y^2-z^2$. Then (1) becomes $\emptyset(u,v) = 0$......(2)

Differentiate partially w.r.to x and y

$\frac{\partial u}{\partial x} = 1 + \frac{\partial z}{\partial x} = 1 + p, \quad \frac{\partial v}{\partial x} = 2x - 2z\frac{\partial z}{\partial x} = 2(x - zp),$

$\frac{\partial u}{\partial y} = 1 + \frac{\partial z}{\partial y} = 1 + q, \quad \frac{\partial v}{\partial y} = 2y - 2z\frac{\partial z}{\partial y} = 2(y - zq).$

Differentiating (2) w.r.t x and y (by applying chain rule) we get,

$\frac{\partial \emptyset}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \emptyset}{\partial v}\frac{\partial v}{\partial x} = 0 \quad or \quad \frac{\partial \emptyset}{\partial u}\frac{\partial u}{\partial x} = -\frac{\partial \emptyset}{\partial v}\frac{\partial v}{\partial x}$.........(3)

$\frac{\partial \emptyset}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \emptyset}{\partial v}\frac{\partial v}{\partial y} = 0 \quad or \quad \frac{\partial \emptyset}{\partial u}\frac{\partial u}{\partial y} = -\frac{\partial \emptyset}{\partial v}\frac{\partial v}{\partial y}$.........(4)

$Dividing\ (3)\ by\ (4), we\ get,\ \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}. \quad \therefore \quad \frac{1+p}{1+q} = \frac{2(x-zp)}{2(y-zq)}.$

$\therefore\ (1+p)(y-zq) = (1+q)(x-zp). \quad \therefore \quad y - zq + py - pqz = x - zp + qx - pqz.$

$\therefore \quad py + pz - qx - qz = x - y. \quad \therefore \quad p(y+z) - q(x+z) = x - y$ is the required PDE.

17. Form the PDE by eliminating the arbitrary function from $\emptyset(xy+z^2, x+y+z) = 0$.

Solution:

Given $\emptyset(\boldsymbol{xy+z^2},\ \boldsymbol{x+y+z}) = \mathbf{0}$.......(1).

Let $u = xy + z^2$ and $v = x + y + z$. Then (1) becomes $\emptyset(u, v) = 0$......(2)

Differentiate partially w.r.to x and y.

$$\frac{\partial u}{\partial x} = y + 2z\frac{\partial z}{\partial x} = y + 2zp, \quad \frac{\partial v}{\partial x} = 1 + \frac{\partial z}{\partial x} = 1 + p,$$

$$\frac{\partial u}{\partial y} = x + 2z\frac{\partial z}{\partial y} = x + 2zq, \quad \frac{\partial v}{\partial y} = 1 + \frac{\partial z}{\partial y} = 1 + q.$$

Differentiating (1) w.r.t x and y, we get,

$$\frac{\partial \emptyset}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \emptyset}{\partial v}\frac{\partial v}{\partial x} = 0 \quad or \quad \frac{\partial \emptyset}{\partial u}\frac{\partial u}{\partial x} = -\frac{\partial \emptyset}{\partial v}\frac{\partial v}{\partial x} \text{.........(3)}$$

$$\frac{\partial \emptyset}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \emptyset}{\partial v}\frac{\partial v}{\partial y} = 0 \quad or \quad \frac{\partial \emptyset}{\partial u}\frac{\partial u}{\partial y} = -\frac{\partial \emptyset}{\partial v}\frac{\partial v}{\partial y} \text{.........(4).}$$

$$Dividing\ (3)\ by\ (4), we\ get,\ \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}. \quad \therefore\ \frac{1+p}{1+q} = \frac{y+2zp}{x+2zq}.$$

$\therefore\ (1 + p)(x + 2zq) = (1 + q)(y + 2zp).$

$\therefore\ x + 2zq + px + 2pqz = y + 2zp + qy + 2pqz.$

$\therefore\ p(x - 2z) - q(y - 2z) + (x - y) = 0$ is the required PDE.

18. Form the PDE by eliminating the arbitrary function from $f(x^2 + y^2, z - xy) = 0$

Solution:

Given $\boldsymbol{f(x^2 + y^2,\ z - xy) = 0}$.......(1).

Let $u = x^2 + y^2$ and $v = z - xy$. Then (1) becomes $\emptyset(u, v) = 0$......(2)

Differentiate partially w.r.to x and y.

$$\therefore\ \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = -y + \frac{\partial z}{\partial x} = -y + p, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial y} = -x + \frac{\partial z}{\partial y} = -x + q.$$

Differentiating (1) w.r.t x and y, we get,

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0 \quad or \quad \frac{\partial \emptyset}{\partial u}\frac{\partial u}{\partial x} = -\frac{\partial}{\partial v}\frac{\partial v}{\partial x} \text{.........(3)}$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 \quad or \quad \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} = -\frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \text{.........(4)}$$

$$Dividing\ (3)\ by\ (4),\ we\ get,\ \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}. \quad \therefore\ \frac{2x}{2y} = \frac{p-y}{q-x}. \quad \therefore\ x(q - x) = y(p - y).$$

$\therefore\ xq - x^2 = yp - y^2. \quad \therefore\ y^2 - x^2 - yp + xq = 0$ is the required PDE.

HOME WORK:

1. Form the PDE by eliminating the arbitrary function from $v = \frac{1}{r}[f(r - at) + g(r - at)]$.

Homogeneous PDE:

If each term of the PDE contains either the dependent variable or one of its partial derivatives, then it is called a homogeneous PDE. Otherwise it is said to be non-homogeneous PDE.

Solution of a non-homogeneous P.D.E by direct integration method:

In this method we find the dependent variable which being the solution, by removing the differential operator through the process of anti-differentiation, i.e., integration.

Problems:

1. Solve $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a.$

Solution:

The given PDE can be rewritten as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{x}{y} + a.$

Integrating w. r. t. x treating y as constant, we get,

$$\frac{\partial z}{\partial y} = \int \left(\frac{x}{y} + a\right) dx + f(y) = \frac{1}{y}\int x dx + a\int 1 dx + f(y).$$

$$\therefore \frac{\partial z}{\partial y} = \frac{x^2}{2y} + ax + f(y).$$ Integrating w. r. t. y, we get,

$$z = \frac{x^2}{2}\int \frac{1}{y} dy + ax\int 1 dy + \int f(y)\, dy + g(x).$$

Thus the solution is $z = \frac{x^2}{2}\log y + axy + F(y) + g(x),$ where $F(y) = \int f(y)\, dy.$

2. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y).$

Solution:

The given PDE can be rewritten as $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\right) = \cos(2x + 3y).$

Integrating w.r.t x treating y as constant, we get,

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \int \cos(2x + 3y)\, dx + f(y).$ $\therefore \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\sin(2x+3y)}{2} + f(y).$

Again integrating w.r.t x.

$$\therefore \frac{\partial z}{\partial y} = \frac{1}{2}\int \sin(2x + 3y)\, dx + f(y)\int 1 dx + g(y).$$

$\therefore \frac{\partial z}{\partial y} = \frac{-\cos(2x+3y)}{4} + xf(y) + g(y).$ Finally integrating w.r.t y, we get,

$z = -\frac{1}{4}\int \cos(2x + 3y)\, dy + x\int f(y)dy + \int g(y)\, dy + h(x).$

Thus the solution is $z = -\frac{1}{12}\sin(2x + 3y) + xF(y) + G(y) + h(x),$

where $F(y) = \int f(y)\, dy, \quad G(y) = \int g(y)\, dy.$

3. Solve $\frac{\partial^2 z}{\partial y^2} = \sin(xy).$

Solution:

The given PDE can be rewritten as $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = sin(xy).$

Integrating w.r.t y by treating x as constant, we get,

$\frac{\partial z}{\partial y} = \int sin(xy)\, dy + f(x) = -\frac{\cos(xy)}{x} + f(x).$

Integrating again w.r.t. y, we get,

$z = \int \left(\frac{-\cos xy}{x} + f(x)\right) dy + g(x) = -\frac{\sin xy}{x^2} + yf(x) + g(x).$

Thus the solution is $u = -\frac{\sin xy}{x^2} + yf(x) + g(x).$ Where f(x) and g(x) are arbitrary functions.

4. Solve $\frac{\partial^2 z}{\partial x^2} = xy$ subjected to the condition that $\frac{\partial z}{\partial x} = \log(1 + y)$ when x=1 and z=0 when x=0.

Solution:

The given PDE can be rewritten as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = xy.$

Integrating w.r.t. x by treating y as constant, we get,

$\frac{\partial z}{\partial x} = \int xy\, dx + f(y).$ $\quad\therefore\quad \frac{\partial z}{\partial x} = \frac{x^2 y}{2} + f(y)$.........(1).

Integrating w.r.t. x treating y as constant, we get,

$z = \int \frac{x^2 y}{2} dx + \int f(y)dy + g(y).$ $\quad\therefore\; z = \frac{x^3 y}{6} + xf(y) + g(y)$.......(2).

Using given data $\frac{\partial z}{\partial x} = \log(1 + y)$ $when\; x = 1\; in\; (1),\; we\; get,$

$\log(1 + y) = \frac{y}{2} + f(y).$ $\quad\therefore\quad f(y) = \log(1 + y) - \frac{y}{2}.$

Substituting f(y) in (2), we get, $z = \frac{x^3 y}{6} + xlog\,(1 + y) - \frac{xy}{2} + g(y) \ldots\ldots (3).$

Using another condition $z = 0\; when\; x = 0\; in\; (3)$, we get, $0 = g(y).$

Thus the solution is $z = \frac{x^3 y}{6} + x\left[\log(1 + y) - \frac{y}{2}\right].$

5. Solve $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y}$ subjected to the conditions $\frac{\partial z}{\partial x} = \log_e x$ when $y = 1$ and $z = 0$ when $x = 1$.

Solution:

The given PDE can be rewritten as $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{x}{y}$.

Integrating w.r.t. y by treating x as constant, we get,

$\frac{\partial z}{\partial x} = \int \frac{x}{y} dy + f(x)$. $\therefore$ $\frac{\partial z}{\partial x} = x \log y + f(x)$………(1)

Integrating w.r.t. x by treating y as constant, we get,

$z = \int x log\, y dx + \int f(x) dx + g(y)$.

$\therefore$ $z = \frac{x^2 log y}{2} + F(x) + g(y)$…….(2). Where $F(x) = \int f(x) dx$.

Using given data $\frac{\partial z}{\partial x} = \log_e x$ $when\ y = 1\ in\ (1),\ we\ get,\ \log_e x = f(x)$.

$\therefore$ $F(x) = \int f(x) dx = \int \log_e x\, dx = xlog\, x - x$. $(Integration\ by\ parts)$.

Substituting F(x) in (2), we get, $z = \frac{x^2 \log y}{2} + x(\log x - 1) + g(y) \ldots \ldots (3)$.

Using another condition $z = 0$ $when\ x = 1\ in\ (3)$, we get, $0 = \frac{logy}{2} - 1 + g(y)$.

$\therefore$ $g(y) = 1 - \frac{logy}{2}$. $Hence\ g(y) = 1 - \log\sqrt{y}$.

Thus the solution is $z = \frac{x^2 \log y}{2} + x(\log x - 1) + 1 - \log\sqrt{y}$.

6. Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ for which $\frac{\partial z}{\partial y} = -2\sin y$ when $x = 0$ and $z = 0$ if y is an odd multiple of $\frac{\pi}{2}$ $\left[\text{or } z = 0 \text{ if } y = (2n+1)\frac{\pi}{2}\right]$.

Solution:

The given PDE can be rewritten as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x \sin y$.

Integrating w. r. t. x teating y as constant, we get,

$\frac{\partial z}{\partial y} = \sin y \int \sin x\, dx + f(y)$. $\therefore$ $\frac{\partial z}{\partial y} = -\sin y \cos x + f(y) \ldots\ldots\ldots (1)$

Integrating w. r. t. **y** treating x as constant, we get,

$z = -\cos x \int \sin y dy + \int f(y) dy + g(x)$.

$\therefore$ $z = (-\cos x)(-\cos y) + F(y) + g(x)$. Where $F(y)=\int f(y)dy$.

Thus, $z = \cos x \cos y + F(y) + g(x) \ldots\ldots\ldots (2)$

By data, $\frac{\partial z}{\partial y} = -2\sin y$ when $x = 0$, using this in (1).

$-2\sin y = (-\sin y).1 + f(y)$. $\therefore$ $f(y) = -\sin y$.

Hence, $F(y) = \int f(y) dy = \int -\sin y dy = \cos y$.

With this (2) becomes $z = \cos x \cos y + \cos y + g(x) \ldots\ldots\ldots (3)$.

Using the given condition $z = 0$ if $y = (2n+1)\frac{\pi}{2}$ in (3), we have

$0 = \cos x \cos(2n+1)\frac{\pi}{2} + \cos(2n+1)\frac{\pi}{2} + g(x)$.

But, $\cos(2n+1)\frac{\pi}{2} = 0$ and hence $0 = 0 + 0 + g(x)$. $\therefore$ $g(x) = 0$.

Thus the solution is $z = \cos x \cos y + \cos y$.

7. Solve $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$ given that $u = 0$ when $t = 0$ and $\frac{\partial u}{\partial t} = 0$ at $x = 0$. Also show that $u \to \sin x$ as $t \to \infty$.

Solution:

The given PDE can be rewritten as $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = e^{-t}\cos x$.

Integrating w r t x by treating t as constant, we get,

$\frac{\partial u}{\partial t} = e^{-t}\sin x + f(t) \ldots\ldots (1)$.

Integrating w.r.t. **t** treating x as constant, we get,

$u = \sin x \int e^{-t}\, dt + \int f(t)\, dt + g(x)$.

$u = -\sin x\, e^{-t} + F(t) + g(x) \ldots\ldots\ldots (2)$. Where $F(t) = \int f(t)\, dt$

By data $\frac{\partial u}{\partial t} = 0$ when $x = 0$. Using this in (1), we get,

$0 = e^{-t}\sin 0 + f(t)$ and hence $f(t) = 0$. $\therefore$ $F(t) = \int f(t)\, dt = \int 0\, dt = 0$.

Substituting $F(t) = 0$, in (2), we get, $u = -\sin x . e^{-t} + g(x) \ldots\ldots\ldots\ldots (3)$.

Also by data, $u = 0$ when $t = 0$. Using this in (3), we get,

$0 = -\sin x\, e^{0} + g(x)$. $\therefore$ $g(x) = \sin x$. Substitute in (3).

$\therefore$ The solution is $u = -e^{-t}\sin x + \sin x$.

Also, $t \to \infty$, W.K.T $e^{-t} \to 0$. Hence $u \to \sin x$ as $t \to \infty$.

HOME WORK:

1. Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos y$ for which $\frac{\partial z}{\partial y} = -2\cos y$ when $x = 0$ and $z = 0$ if $y = n\pi$. n is an integer.

2. Solve $\frac{\partial^2 z}{\partial x \partial t} = e^{-2t}\cos 3x$ given that $z = 0$ when $t = 0$ and $\frac{\partial z}{\partial t} = 0$ at $x = 0$.

3. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0$.

Solutions of homogeneous PDE involving derivative w. r. t. one variable:

Problems:

1. solve the equation $\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}-6z=0$ given that $z=x$ and $\frac{\partial z}{\partial y}=0$, when $y=0$.

Solution:

Here z is a function of y only. The given PDE can assumes the form of ODE as

$\frac{d^2z}{dy^2}+\frac{dz}{dy}-6z=0.\quad \therefore\ (D^2+D-6)z=0.\quad Where\ \frac{d}{dy}=D.$

$The\ AE\ is\ given\ by\ \ m^2+m-6=0.\ \therefore\ m^2+3m-2m-6=0.$

$\therefore m(m+3)-2(m+3)=0.\quad \therefore\ (m+3)(m-2)=0.\quad \therefore\ m=2,-3.$

The solution of ODE is given by $z=c_1e^{2y}+c_2e^{-3y}$.

Solution of PDE can be obtained by replacing $c_1\ and\ c_2$ by functions of x.

Hence the solution of the PDE is given by $z=f(x)e^{2y}+g(x)e^{-3y}$........(1)

Now we shall make use of given conditions in order to find f(x) and g(x)

Given when y=0, $z=x$. Using these conditions in (1), we get,

$x=f(x)+g(x)\ldots\ldots(2).$

Differentiating (1) partially w. r .t. y, we get, $\frac{\partial z}{\partial y}=2f(x)e^{2y}-3g(x)e^{-3y}$..........(3).

Further using other condition i.e when y=0, $\frac{\partial z}{\partial y}=0$ in (3), we get,

$0=2f(x)-3g(x).\quad \therefore\ f(x)=\frac{3}{2}g(x)$......(4).

$Solving\ eq(3)\ and\ eq(4)$, we get, $f(x)=\frac{3x}{5}$ and $g(x)=\frac{2x}{5}$.

Substituting f(x) and g(x) in (1), we get,

$z=\frac{x}{5}(3e^{2y}+2e^{-3y})$ is the required solution.

2. Solve the equation $\frac{\partial^2 z}{\partial y^2}-4\frac{\partial z}{\partial y}+4z=0$ given that $z=0$ and $\frac{\partial z}{\partial y}=e^x$, when $y=0$.

Solution:

Here z is a function of y only. The given PDE can assumes the form of ODE as

$\frac{d^2z}{dy^2}-4\frac{dz}{dy}+4z=0.\ \therefore\ (D^2-4D+4)z=0.\quad Where\ \frac{d}{dy}=D.$

$The\ AE\ is\ given\ by\ \ m^2 - 4m + 4 = 0. \quad \therefore\ (m-2)^2 = 0. \quad \therefore \quad m = 2, 2.$

The solution of ODE is given by $z = e^{2y}(c_1 + c_2 y)$.

$\therefore$ Solution of PDE can be obtained by replacing $c_1 and\ c_2$by functions of x.

Hence the solution of the PDE is given by $z = e^{2y}[f(x) + g(x)y]$.......(1).

Now we shall make use of given conditions in order to find f(x) and g(x).

Given when y=0, $z = 0$. Using above conditions in (1), we get, $0 = f(x)$.

Differentiating (1) partially w.r.t. y, we get,

$\frac{\partial z}{\partial y} = 2e^{2y}f(x) + g(x)[2ye^{2y} + e^{2y}]$ ....(2) .

Further using other condition i.e when y=0, $\frac{\partial z}{\partial y} = e^x$ in eq(2)

$e^x = 2f(x) + g(x). \ \therefore\ g(x) = e^x.$

Substituting f(x) and g(x) in (1), we get,

$z = e^{2y}[e^x y]. \quad \therefore \quad z = ye^{x+2y}$ is the required solution.

3. Solve the equation $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial z}{\partial x} - 4z = 0$ given that $z = 1$ and $\frac{\partial z}{\partial x} = y$, when $x = 0$.

Solution:

Here z is a function of x only. The given PDE can assumes the form of ODE as

$\frac{d^2z}{dx^2} + 3\frac{dz}{dx} - 4z = 0. \ \therefore\ (D^2 + 3D - 4)z = 0. \quad Where \frac{d}{dx} = D.$

$The\ AE\ is\ given\ by\ \ m^2 + 3m - 4 = 0. \quad \therefore\ m^2 + 4m - m - 4 = 0.$

$\therefore\ m(m+4) - 1(m+4) = 0. \quad \therefore\ (m+4)(m-1) = 0. \quad \therefore\ m = 1, -4.$

The solution of ODE is given by $z = c_1 e^x + c_2 e^{-4x}$.

Solution of PDE can be obtained by replacing $c_1 and\ c_2$by functions of y.

Hence the solution of the PDE is given by $z = f(y)e^x + g(y)\ e^{-4x}$.......(1).

Now we shall make use of given conditions in order to find f(y) and g(y).

Given when x=0, $z = 1$.

Using above conditions in (1), we get, $1 = f(y) + g(y)$.......(2).

Differentiating (1) partially w. r .t. x, we get, $\frac{\partial z}{\partial x} = e^x f(y) - 4e^{-4x}g(y)$.......(3).

Further using other condition i.e when x=0, $\frac{\partial z}{\partial x}$ =y in (3), we get,

$1 = f(y) - 4g(y)$……(4). Solving (3) and (4), we get, $f(y) = \frac{1}{5}(y+4)$ and $g(y) = \frac{1}{5}(1-y)$.

Substituting f(y) and g(y) in (1), we get,

$z = \frac{1}{5}(4+y)e^{x} + \frac{1}{5}(1-y)e^{-4x}$ is the required solution.

4. Solve the equation $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + 2z = 0$ given that $z = e^y$ and $\frac{\partial z}{\partial x} = 0$, when $x = 0$.

Solution:

Here z is a function of x only. The given PDE can assumes the form of ODE as

$\frac{d^2z}{dx^2} - 2\frac{dz}{dx} + 2z = 0. \quad \therefore \quad (D^2 - 2D + 2)z = 0. \qquad Where\ \frac{d}{dx} = D.$

$The\ AE\ is\ given\ by\ m^2 - 2m + 2 = 0. \quad \therefore \quad m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$

The solution of ODE is given by $z = e^{x}(c_1 \cos x + c_2 \sin x)$

Solution of PDE can be obtained by replacing $c_1 and\ c_2$ by functions of y

Hence the solution of the PDE is given by

$z = e^{x}[f(y)\cos x + g(y)\sin x]$……..(1)

Now we shall make use of given conditions in order to find f(y) and g(y)

Given when x=0 , $z = e^{y}$

Using above conditions in (1), we get, $e^{y} = f(y)$.

Differentiating (1) partially w. r .to x, we get

$\frac{\partial z}{\partial x} = e^{x}[-\sin x f(y) + cosx\ g(y)] + e^{x}[f(y)\cos x + g(y)\sin x]$……….(2)

Further using other condition i.e when x=0, $\frac{\partial z}{\partial x} = 0$ in (2), we get,

$0 = g(y) + f(y). \ \therefore \ g(y) = -e^{y}.$

Substituting f(y) and g(y) in (1), we get,

$z = e^{x}e^{y}[cosx - \sin x]$ is the required solution.

5. Solve the equation $\frac{\partial^3 u}{\partial y^3} + 4\frac{\partial u}{\partial y} = 0$ given that $u = 0$, $\frac{\partial u}{\partial y} = x$, $\frac{\partial^2 u}{\partial y^2} = x^2 - 1$, when $y = 0$.

Solution:

Here u is a function of y only. The given PDE can assumes the form of ODE as

$\frac{d^3u}{dy^3} + 4\frac{du}{dy} = 0. \quad \therefore \ (D^3 + 4D)u = 0. \quad Where \frac{d}{dy} = D.$

$The\ AE\ is\ given\ by\ \ m^3 + 4m = 0. \quad \therefore \quad m(m^2 + 4) = 0. \quad \therefore \quad m = 0, \ \pm 2i.$

The solution of ODE is given by $u = c_1 + c_2 \cos 2y + c_3 \sin 2\, y).$

Hence the solution of the PDE is given by $u = f(x) + g(x) \cos 2y + h(x) \sin 2y$.......(1)

Now we shall make use of given conditions in order to find f(x), g(x) and h(x).

Using given condition u=0 when y=0 in (1), we get, $0 = f(x) + g(x).$

Differentiating (1) partially w. r .t. y, we get $\frac{\partial u}{\partial y} = -2 \sin 2y\ g(x) + 2 \cos 2y\ h(x)$.......(2).

Using given condition i.e when y=0 , $\frac{\partial u}{\partial y} = x$ in (2), we get, $x = 2h(x). \quad \therefore \ h(x) = \frac{x}{2}.$

Differentiating (2) partially w. r .t. y, we get $\frac{\partial^2 u}{\partial y^2} = -4 \cos 2y\ g(x) - 4 \sin 2y\ h(x)$....(3).

Using given condition i.e when y=0, $\frac{\partial^2 u}{\partial y^2} = x^2 - 1$ in (3), $x^2 - 1 = -4\, g(x).$

$\therefore \ g(x) = \frac{1-x^2}{4}. \quad \therefore \ f(x) = \frac{x^2-1}{4}.$

Substituting f(x), g(x) and h(x) in (1) , we get,

$z = \frac{x^2-1}{4} + \left(\frac{1-x^2}{4}\right) \cos 2y + \frac{x \sin 2y}{2}$ is the required solution.

6. Solve the equation $\frac{\partial^2 z}{\partial x^2} + z = 0$ given that $x = 0, \ z = e^y$ and $\frac{\partial z}{\partial x} = 1.$

Solution:

Here z is a function of x only. The given PDE can assumes the form of ODE as

$\frac{d^2z}{dx^2} + z = 0. \quad \therefore \ (D^2 + 1)z = 0. \quad Where \frac{d}{dx} = D.$

$The\ AE\ is\ given\ by\ m^2 + 1 = 0. \quad \therefore \ m = \pm i.$

The solution of ODE is given by $z = c_1 \cos x + c_2 \sin x.$

Solution of PDE can be obtained by replacing $c_1\ and\ c_2$ by functions of y.

Hence the solution of the PDE is given by $z = f(y) \cos x + g(y) \sin x$......(1).

Now we shall make use of given conditions in order to find f(y) and g(y).

Given when x=0, $z = e^y.$

Using above conditions in (1), we get, $e^y = f(y).$

Differentiating (1) partially w. r .t. x, we get $\frac{\partial z}{\partial x} = -\sin x f(y) + cosx\ g(y)$.......(2)

Further using other condition i.e when x=0, $\frac{\partial z}{\partial x} = 1$ in (2), we get, $1 = g(y)$.

Substituting f(y) and g(y) in (1), we get,

$z = e^y \cos x + \sin x$ is the required solution.

7. Solve the equation $\frac{\partial^2 z}{\partial y^2} + z = 0$ given that $z = \cos x$ and $\frac{\partial z}{\partial y} = \sin x$ when $y = 0$.

Solution:

Here z is a function of y only. The given PDE can assumes the form of ODE as

$\frac{d^2 z}{dy^2} + z = 0$. $\therefore$ $(D^2 + 1)z = 0$. $Where\ \frac{d}{dy} = D$.

$The\ AE\ is\ given\ by\ m^2 + 1 = 0$. $\therefore$ $m = \pm i$.

The solution of ODE is given by $z = c_1 \cos y + c_2 \sin y$.

Solution of PDE can be obtained by replacing $c_1 and\ c_2$by functions of x

Hence the solution of the PDE is given by $z = f(x) \cos y + g(x) siny$.......(1)

Now we shall make use of given conditions in order to find f(x) and g(x).

Given when y=0, $z = cosx$.

Using above conditions in (1), we get, $cosx = f(x)$.

Differentiating (1) partially w. r .t. y, we get $\frac{\partial z}{\partial y} = -\sin y f(x) + cosy\ g(x)$.......(2)

Further using other condition i.e when y=0, $\frac{\partial z}{\partial y} = \sin x$ in (2), we get,

$\sin x = g(x)$ Substituting f(x) and g(x) in (1), we get,

$z = \cos x\ cosy + \sin x sin\ y = \cos(x - y)$ is the required solution.

8. Solve the equation $\frac{\partial^2 z}{\partial y^2} = z$ given that $y = 0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$.

Solution:

Here z is a function of y only. The given PDE can assumes the form of ODE as

$\frac{d^2 z}{dy^2} - z = 0$. $\therefore$ $(D^2 - 1)z = 0$. $Where\ \frac{d}{dy} = D$.

$The\ AE\ is\ given\ by\ m^2 - 1 = 0$. $\therefore$ $m = \pm 1$.

The solution of ODE is given by $z = c_1 e^y + c_2 e^{-y}$.

Solution of PDE can be obtained by replacing c_1 and c_2 by functions of x.

Hence the solution of the PDE is given by $z = f(x)e^y + g(x)e^{-y}$.......(1).

Now we shall make use of given conditions in order to find f(x) and g(x).

Given when y=0, $z = e^x$.

Using above conditions in (1), we get, $e^x = f(x) + g(x) \ldots \ldots (2)$.

Differentiating (1) partially w. r .t. y, we get $\frac{\partial z}{\partial y} = e^y f(x) - e^{-y} g(x)$..........(3).

Further using other condition i.e when y=0 , $\frac{\partial z}{\partial y} = e^{-x}$ in (3).

$e^{-x} = f(x) - g(x) \ldots\ldots (4).$ $Solving (2) and \ (4), \ we \ get,$

$f(x) = \frac{e^x + e^{-x}}{2} = \cos hx. \quad \therefore \quad g(x) = \frac{e^x - e^{-x}}{2} = \sin hx.$

Substituting f(x) and g(x) in (1), we get,

$z = e^y \cos hx + e^{-y} \sin hx$ is the required solution.

9. Solve the equation $\frac{\partial^2 z}{\partial x^2} = a^2 z$ given that $x = 0$, $z = 0$ and $\frac{\partial z}{\partial x} = asiny$.

Solution:

Here z is a function of x only. The given PDE can assumes the form of ODE as

$\frac{d^2 z}{dx^2} - a^2 z = 0. \quad \therefore \ (D^2 - a^2)z = 0. \quad Where \frac{d}{dx} = D.$

$The \ AE \ is \ given \ by \ m^2 - a^2 = 0. \quad \therefore \ m = \pm a.$

The solution of ODE is given by $z = c_1 e^{ax} + c_2 e^{-ax}$.

Solution of PDE can be obtained by replacing $c_1 and \ c_2$ by functions of y

Hence the solution of the PDE is given by $z = f(y)e^{ax} + g(y)e^{-ax}$.......(1)

Now we shall make use of given conditions in order to find f(y) and g(y)

Given when x=0, $z = 0$.

Using above conditions in (1), we get, $0 = f(y) + g(y). \quad \therefore \ f(y) = -g(y).$

Differentiating (1) partially w. r .t. x, we get,

$\frac{\partial z}{\partial x} = ae^{ax} f(y) - ae^{-ax} g(y)$..........(2)

Further using other condition i.e when x=0 , $\frac{\partial z}{\partial x} = asiny$ in (2), we get,

$a\sin y = af(y) - ag(y)$. $\therefore$ $a\sin y = 2a\, f(y)$. $\therefore$ $f(y) = \frac{siny}{2}$. $\therefore$ $g(y) = -\frac{siny}{2}$.

Substituting f(y) and g(y) in (1), we get, $z = \frac{1}{2}(e^{ax}\sin y - e^{-ax}siny)$.

$\therefore$ $z = \sin y\; sin\,h\; ax$ is the required solution.

HOME WORK:

1. Solve the equation $\frac{\partial^2 z}{\partial x^2} + 4z = 0$ given that when $x = 0$, $z = e^{2y}$ and $\frac{\partial z}{\partial x} = 2$.

2. Solve the equation $\frac{\partial^2 z}{\partial x^2} - 16z = 0$ given that when $x = 0$, $z = 0$ and $\frac{\partial z}{\partial x} = 4\sin y$.

Method of separation of variables:

It involves a solution which breaks up into a product of functions each of which contains only of the variables. The following examples explains this method.

1. Solve by the method of separation of variables $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where u(x,0)=$6e^{-3x}$

Solution:

Given $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ .... (1).

Let u= XT.......(2). Where X=X(x) and T=T(t) be the solution of the given PDE.

Substituting above equations into the given PDE

$\therefore$ $\frac{\partial(XT)}{\partial x} = 2\frac{\partial(XT)}{\partial t} + XT$. $\therefore$ $T\frac{dX}{dx} = 2X\frac{dT}{dt} + XT$. Separate the variables.

Dividing above equation throughout by XT,

$\frac{1}{X}\frac{dX}{dx} = 2\frac{1}{T}\frac{dT}{dt} + 1$. $\therefore$ $\frac{1}{X}\frac{dX}{dx} = \frac{2}{T}\frac{dT}{dt} + 1$.

Equating both sides of above equation to a common constant k, we have

$\frac{1}{X}\frac{dX}{dx} = k$; $\frac{2}{T}\frac{dT}{dt} + 1 = k$.

$\frac{1}{X}\frac{dX}{dx} = k$; $\frac{2}{T}\frac{dT}{dt} = k - 1$.

$\frac{dX}{X} = (k)dx$; $\frac{dT}{T} = \frac{(k-1)}{2}dt$.

Integrating above equations , we get,

$\int\frac{dX}{X} = \int(k)\,dx$; $\int\frac{dT}{T} = \int\frac{(k-1)}{2}dt$.

$\log X = kx + c_1$; $\log T = \frac{(k-1)t}{2} + c_2$.

$X = e^{kx+c_1}$; $Y = e^{\frac{(k-1)t}{2}+c_2}$.

$u = XY = e^{c_1+c_2}\, e^{kx+\frac{(k-1)t}{2}}$. $Let \quad c = e^{c_1+c_2}$.

$Thus \quad u = c\, e^{kx+\frac{(k-1)t}{2}}$ .... (3) is the required solution.

Given u(x,0) =$6e^{-3x}$. Using in (3), $6e^{-3x} = ce^{kx}$. $\therefore\ c = 6\,, k = -3$

$\therefore \quad u = 6\, e^{-3x-2t}$ is the required solution.

2. Solve by the method of separation of variables $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ where u(0,y)=$2e^{5y}$.

Solution:

Given $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$........... (1)

Let u= XY......... (2). Where X=X(x) and Y=Y(y) be the solution of the given PDE.

Substituting above equations into the given PDE,

$4\frac{\partial (XY)}{\partial x} + \frac{\partial (XY)}{\partial y} = 3(XY)$. $\therefore\ 4Y\frac{dX}{dx} + X\frac{dY}{dy} = 3XY$.

Dividing above equation throughout by XY,

$\frac{4}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 3$. $\therefore\ \frac{4}{X}\frac{dX}{dx} = -\frac{1}{Y}\frac{dY}{dy} + 3$.

Equating both sides of above equation to a common constant k, we have,

$\frac{4}{X}\frac{dX}{dx} = k$; $-\frac{1}{Y}\frac{dY}{dy} + 3 = k$.

$\frac{1}{X}\frac{dX}{dx} = \frac{k}{4}$; $\frac{1}{Y}\frac{dY}{dy} = 3 - k$.

$\frac{dX}{X} = \frac{k}{4}dx$; $\frac{dY}{Y} = (3-k)dy$.

Integrating above equations, we get,

$\int \frac{dX}{X} = \int \left(\frac{k}{4}\right) dx$; $\int \frac{dY}{Y} = \int (3-k)\, dy$.

$\log X = \frac{k}{4}x + c_1$; $\log Y = (3-k)y + c_2$.

$X = e^{\frac{kx}{4}+c_1}$; $Y = e^{(3-k)y+c_2}$.

$\therefore\ u = XY = e^{c_1+c_2}\, e^{\frac{kx}{4}+(3-k)y}$. $Let \quad c = e^{c_1+c_2}$.

$Thus \quad u = c\, e^{\frac{kx}{4}+(3-k)y}$ ... ... (3) is the required solution.

$Given\, u(0,y) = 2e^{5y}$. Using this in (3), $2e^{5y} = c^{(3-k)y}$.

$\therefore \quad c = 2$ and $5 = 3 - k,\ k = -2$.

$\therefore \quad u = 2e^{-\frac{x}{2}+5y}$ is the required solution.

3. Solve by the method of separation of variables $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} = 0$.

Solution:

Given $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} = 0$ ………. (1).

Let u= XY……… (2). Where X=X(x) and Y=Y(y) be the solution of the given PDE.

Substituting above equations into the given PDE,

$x^2\frac{\partial(XY)}{\partial x} + y^2\frac{\partial(XY)}{\partial y} = 0. \quad \therefore \ x^2Y\frac{dX}{dx} + Xy^2\frac{dY}{dy} = 0.$

Dividing above equation throughout by XY,

$\frac{x^2}{X}\frac{dX}{dx} + \frac{y^2}{Y}\frac{dY}{dy} = 0. \qquad \therefore \ \frac{x^2}{X}\frac{dX}{dx} = -\frac{y^2}{Y}\frac{dY}{dy}.$

Equating both sides of above equation to a common constant k, we have,

$\frac{x^2}{X}\frac{dX}{dx} = k \quad ; \quad -\frac{y^2}{Y}\frac{dY}{dy} = k.$

$\frac{1}{X}\frac{dX}{dx} = \frac{k}{x^2} \quad ; \quad \frac{1}{Y}\frac{dY}{dy} = -\frac{k}{y^2}.$

$\frac{dX}{X} = \left(\frac{k}{x^2}\right)dx \quad ; \quad \frac{dY}{Y} = \left(-\frac{k}{y^2}\right)dy.$

Integrating above equations, we get,

$\int\frac{dX}{X} = \int\left(\frac{k}{x^2}\right)dx \quad ; \quad \int\frac{dY}{Y} = -\int\left(\frac{k}{y^2}\right)dy.$

$\log X = -\frac{k}{x} + c_1 \quad ; \quad \log Y = \frac{k}{y} + c_2.$

$X = e^{-\frac{k}{x}+c_1} \quad ; \quad Y = e^{\frac{k}{y}+c_2}.$

$\therefore \quad u = XY = e^{c_1+c_2}\, e^{k\left(\frac{1}{y}-\frac{1}{x}\right)}. \quad Let \quad c = e^{c_1+c_2}.$

$Thus \quad u = c\, e^{k\left(\frac{1}{y}-\frac{1}{x}\right)}$ is the required solution.

4. Solve by the method of separation of variables $py^3 + qx^2 = 0$.

Solution:

Given $py^3 + qx^2 = 0$.........(1). Where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. $\therefore$ $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0$.

Let z= XY .... (2). Where X=X(x) and Y=Y(y) be the solution of the given PDE.

Substituting above equations into the given PDE,

$$y^3 \frac{\partial(XY)}{\partial x} + x^2 \frac{\partial(XY)}{\partial y} = 0. \quad \therefore \; Yy^3 \frac{dX}{dx} + Xx^2 \frac{dY}{dy} = 0.$$

Dividing above equation throughout by XY,

$$\frac{y^3}{X} \frac{dX}{dx} + \frac{x^2}{Y} \frac{dY}{dy} = 0. \quad \therefore \; \frac{y^3}{X} \frac{dX}{dx} = -\frac{x^2}{Y} \frac{dY}{dy}.$$

Equating both sides of above equation to a common constant k, we have,

$$\frac{1}{x^2 X} \frac{dX}{dx} = k \quad ; \quad -\frac{1}{y^3 Y} \frac{dY}{dy} = k.$$

$$\frac{1}{X} \frac{dX}{dx} = kx^2 \quad ; \quad \frac{1}{Y} \frac{dY}{dy} = -ky^3.$$

$$\frac{dX}{X} = (kx^2)dx \quad ; \quad \frac{dY}{Y} = (-ky^3)dy.$$

Integrating above equations , we get,

$$\int \frac{dX}{X} = \int (kx^2)\, dx \quad ; \quad \int \frac{dY}{Y} = \int (-ky^3)\, dy.$$

$$\log X = \frac{kx^3}{3} + c_1 \quad ; \quad \log Y = -\frac{ky^4}{4} + c_2.$$

$$X = e^{\frac{kx^3}{3} + c_1} \quad ; \quad Y = e^{-\frac{ky^4}{4} + c_2}.$$

$$\therefore \; z = XY = e^{c_1 + c_2}\, e^{\frac{kx^3}{3} - \frac{ky^4}{4}}. \quad Let \quad c = e^{c_1 + c_2}.$$

$Thus \quad z = c\, e^{k(\frac{x^3}{3} - \frac{y^4}{4})}$ is the required solution.

5. Solve by the method of separation of variables $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x + y)u$.

Solution:

Given $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x + y)u$......... (1)

Let u= XY..........(2). Where X=X(x) and Y=Y(y) be the solution of the given PDE.

Substituting above equation into the given PDE,

$$\frac{\partial(XY)}{\partial x} + \frac{\partial(XY)}{\partial y} = 2(x + y)(XY). \quad \therefore \; Y \frac{dX}{dx} + X \frac{dY}{dy} = 2(x + y)XY.$$

Dividing above equation throughout by XY,

$\frac{1}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 2(x + y).$ $\quad \therefore \quad \frac{1}{X}\frac{dX}{dx} - 2x = -\frac{1}{Y}\frac{dY}{dy} + 2y.$

Equating both sides of above equation to a common constant k, we have,

$\frac{1}{X}\frac{dX}{dx} - 2x = k \quad ; \quad -\frac{1}{Y}\frac{dY}{dy} + 2y = k.$

$\frac{1}{X}\frac{dX}{dx} = 2x + k \quad ; \quad \frac{1}{Y}\frac{dY}{dy} = 2y - k.$

$\frac{dX}{X} = (2x + k)dx \quad ; \quad \frac{dY}{Y} = (2y - k)dy.$

Integrating above equations , we get,

$\int \frac{dX}{X} = \int (2x + k)\,dx \quad ; \quad \int \frac{dY}{Y} = \int (2y - k)\,dy.$

$\log X = x^2 + kx + c_1 \quad ; \quad \log Y = y^2 - ky + c_2.$

$X = e^{x^2+kx+c_1} \quad ; \quad Y = e^{y^2-ky+c_2}.$

$\therefore \quad u = XY = e^{c_1+c_2}\, e^{x^2+y^2+kx-ky}. \quad Let \quad c = e^{c_1+c_2}.$

$Thus \quad u = c\, e^{x^2+y^2+k(x-y)}$ is the required solution.

6. Solve by the method of separation of variables $x^2 \frac{\partial^2 z}{\partial x \partial y} + 3y^2 z = 0$

Solution:

Given $x^2 \frac{\partial^2 z}{\partial x \partial y} + 3y^2 z = 0$.......... (1).

Let z = XY.........(2). Where X=X(x) and Y=Y(y) be the solution of the given PDE.

Differentiating z=XY partially we get, $\frac{\partial z}{\partial x} = Y\frac{dX}{dx} \quad and \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{dX}{dx}\frac{dY}{dy}.$

Substituting above equations into the given PDE,

$x^2 \frac{dX}{dx}\frac{dY}{dy} + 3\,y^2(XY) = 0. \quad \therefore \quad x^2 \frac{1}{X}\frac{dX}{dx} = -3y^2 \frac{Y}{\left(\frac{dY}{dy}\right)}.$

Equating both sides of above equation to a common constant k, we have,

$\frac{1}{X}\frac{dX}{dx} = \frac{k}{x^2} \quad ; \quad -3y^2 \frac{Y}{\left(\frac{dY}{dy}\right)} = k.$

$\frac{1}{X}\frac{dX}{dx} = \frac{k}{x^2} \quad ; \quad k\frac{dY}{dy} = -3y^2 Y.$

$\frac{dX}{X} = \left(\frac{k}{x^2}\right)dx \quad ; \quad \frac{dY}{Y} = \left(-\frac{3y^2}{k}\right)dy.$

Integrating above equations , we get,

$\int \frac{dX}{X} = \int \left(\frac{k}{x^2}\right) dx$; $\int \frac{dY}{Y} = \int -\frac{3y^2}{k} dy.$

$\log X = -\frac{k}{x} + c_1$; $\log Y = -\frac{y^3}{k} + c_2.$

$X = e^{-\frac{k}{x}+c_1}$; $Y = e^{-\frac{y^3}{k}+c_2}.$

We have $z = XY$ $\therefore$ $z = e^{c_1+c_2} e^{-\frac{k}{x}-\frac{y^3}{k}}.$

$\therefore$ $z = c\, e^{-\frac{k}{x}-\frac{y^3}{k}}$ is the required solution. $Where\ c = e^{c_1+c_2}.$

7. Solve by the method of separation of variables $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

Solution:

Given $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.... (1)

Let z= XY ………(2). Where X=X(x) and Y=Y(y) be the solution of the given PDE.

Differentiating z=XY partially we get

$\frac{\partial z}{\partial x} = Y\frac{dX}{dx}$, $\frac{\partial z}{\partial y} = X\frac{dY}{dy}$ and $\frac{\partial^2 z}{\partial x^2} = Y\frac{d^2X}{dx^2}.$

Substituting above equations into the given PDE,

$Y\frac{d^2X}{dx^2} - 2Y\frac{dX}{dx} + X\frac{dY}{dy} = 0.$ Dividing by XY,

$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} = -\frac{1}{Y}\frac{dY}{dy}.$

Equating both sides of above equation to a common constant k, we have,

$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} = k$ and $-\frac{1}{Y}\frac{dY}{dy} = k.$

$\therefore$ $\frac{d^2X}{dx^2} - 2\frac{dX}{dx} - kX = 0$ and $\frac{dY}{dy} + kY = 0.$

Both are linear homogeneous ODE with constants coefficients .

AE's are given by

$m^2 - 2m - k = 0.$ and $m + k = 0.$

$m = 1 \pm \sqrt{1+k}$ and $m = -k.$

$\therefore$ Solution ODE's are

$X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}$ and $Y = c_3 e^{-ky}.$

We have $z = XY$. $\therefore\ z = \left(c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}\right)(c_3 e^{-ky})$.

$\therefore\ z = e^{x-ky}\,(A\,e^{(\sqrt{1+k})x} + Be^{-(\sqrt{1+k})x}$ is the required solution .

Where $c_1c_2 = A\ \ and\ \ c_2c_3 = B$.

8. Solve by the method of separation of variables $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} + 2z$ subjected to the conditions z(0,y)=0 $and\ \frac{\partial z}{\partial x}(0, y) = e^{2y}$.

Solution:

Given $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} + 2z$ ....(1).

Let z= XY.........(2). Where X=X(x) and Y=Y(y) be the solution of the given PDE.

Differentiating z=XY partially we get,

$\frac{\partial z}{\partial x} = Y\frac{dX}{dx}\ ,\frac{\partial z}{\partial y} = X\frac{dY}{dy}\ \ and\ \frac{\partial^2 z}{\partial x^2} = Y\frac{d^2X}{dx^2}$.

Substituting above equations into the given PDE,

$Y\frac{d^2X}{dx^2} - X\frac{dY}{dy} = 2XY$. Dividing by XY. $\frac{1}{X}\frac{d^2X}{dx^2} - \frac{1}{Y}\frac{dY}{dy} = 2$. $\therefore \frac{1}{X}\frac{d^2X}{dx^2} = 2 + \frac{1}{Y}\frac{dY}{dy}$.

Equating both sides of above equation to a common constant k, we have,

$\frac{1}{X}\frac{d^2X}{dx^2} = k \quad and \quad 2 + \frac{1}{Y}\frac{dY}{dy} = k.$

$\therefore\ \frac{d^2X}{dx^2} - kX = 0 \quad and \quad \frac{dY}{dy} - (k-2)Y = 0.$

Both are linear homogeneous ODE with constants coefficients.

AE's are given by

$m^2 - k = 0 \quad and \quad m - (k-2) = 0.$

$m = \pm\sqrt{k}$ and $m = k - 2$.

Solution ODE's are

$X = c_1 e^{(\sqrt{k})x} + c_2 e^{(-\sqrt{k})x}$ and $Y = c_3 e^{(k-2)y}$

We have $z = XY$.

$\therefore\ z = \left(c_1 e^{(\sqrt{k})x} + c_2 e^{(-\sqrt{k})x}\right)\left(c_3 e^{(k-2)y}\right).$

$\therefore\ z = e^{(k-2)y}\,(A\,e^{(\sqrt{k})x} + Be^{-(\sqrt{k})x}$ ... ... (3) is the required solution .

Where $c_1 c_2 = A \; and \; c_2 c_3 = B$

Given $z(0, y) = 0, using \; in \; eq(3), \; 0 = A + B.$ ∴ A= − B

Differentiate (3) w.r.t. x,

$\frac{\partial z}{\partial x} = e^{(k-2)y}(A\sqrt{k}\, e^{(\sqrt{k})x} - B\sqrt{k}\; e^{-(\sqrt{k})x})$......(4).

Using $\frac{\partial z}{\partial x}(0, y) = e^{2y}$ in (4), $e^{2y} = \sqrt{k}e^{(k-2)y}(A - B).$

$\therefore \; k - 2 = 2 \quad and \;\; A - B = \frac{1}{\sqrt{k}}. \quad \therefore \; k = 4. \quad \therefore \; A = \frac{1}{2\sqrt{k}}, \quad B = -\frac{1}{2\sqrt{k}}$.

$\therefore \quad A = \frac{1}{4} \quad and \quad B = -\frac{1}{4}. \quad \therefore \; z = \frac{e^{2y}}{4}(e^{2x} - e^{-2x}).$

$\therefore \quad z = \frac{e^{2y} \sin h 2y}{2}$ is the required solution.

HOME WORK:

1. **Solve by the method of separation of variables $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, where u(0, y)=8$e^{-3y}$.**

3. **Solve by the method of separation of variables $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, where u(x, 0)=4$e^{-x}$.**

Various possible solutions of the one-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ by the method of separation of variables.

Consider $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ --------------(1)

Assume that that the solution of (1) is of the form u = XT, where
X is a function of x and T is a function of t only.

$$\frac{\partial^2(XT)}{\partial t^2} = c^2\frac{\partial^2(XT)}{\partial x^2}$$

$$X\frac{d^2T}{dt^2} = c^2 T\frac{d^2X}{dx^2}$$

Dividing by $c^2 \, XT$

$$\frac{1}{c^2 T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}$$

Equating both sides to a common constant k we have,

$$\frac{1}{X}\frac{d^2X}{dx^2} = k \quad and \quad \frac{1}{c^2 T}\frac{d^2T}{dt^2} = k$$

$$\frac{d^2X}{dx^2} - kX = 0 \quad and \quad \frac{d^2T}{dt^2} - c^2kT = 0$$

$(D^2 - k)X = 0 \quad and \quad (D^2 - c^2k)T = 0$ ------------------(2)

Case (i) : Let k=0

The equation (2) become $D^2X = 0 \quad and \quad D^2T = 0$

In both the equations AE is $m^2 = 0 \quad \therefore m = 0,0$

Solutions are giving by

$$X = (C_1 + C_2x)e^{0x} \quad ; \quad T = (C_3 + C_4t)e^{0t}$$

$$X = (C_1 + C_2x) \quad ; \quad T = (C_3 + C_4t)$$

Hence the solution of the PDE (when constant is 0) is given by

$u = XT = (C_1 + C_2x)(C_3 + C_4t)$

Case (ii): Let k is positive , k = + p^2(say)

The equation (2) become $(D^2 - p^2)X = 0 \quad and \quad (D^2 - c^2p^2)T = 0$

AEs are $m^2 - p^2 = 0$ and $m^2 - c^2p^2 = 0$

$\therefore m = \pm p$ and $\therefore m = \pm cp$

Solutions are giving by

$$X = C_1e^{px} + C_2e^{-px} \quad ; \quad T = C_3e^{cpt} + C_4e^{-cpt}$$

Hence the solution of the PDE (when constant is positive) is given by

$u = XT = (C_1e^{px} + C_2e^{-px})(C_3e^{cpt} + C_4e^{-cpt})$

Case (iii): Let k is negative and k $=-p^2$(say)

The equation (2) become $(D^2 + p^2)X = 0 \quad and \quad (D^2 + c^2p^2)T = 0$

AEs are $m^2 + p^2 = 0$ and $m^2 + c^2p^2 = 0$

$\therefore m = \pm ip$ and $\therefore m = \pm icp$

Solutions are giving by

$$X = C_1 \cos px + C_2 \sin px \quad ; \quad T = C_3 \cos cpt + C_4 \sin cpt$$

Hence the solution of the PDE (when constant is negative) is given by

$u = XT = (C_1 \cos px + C_2 \sin px)C_3 \cos cpt + C_4 \sin cpt)$

Of these three solutions, we have to choose that the solution which is consistent with the physical nature of the problem. As we will be dealing with problem of vibrations, u must be a periodic function of x and t. Hence their solution must involve trigonometric terms. Accordingly the solution given by case (iii) of the form $y = (C_1\text{cospx} + C_2\text{sinpx})\ (C_3\text{coscpt} + C_4\text{sincpt}\ )$is the only suitable solution of the wave equation.

Various possible solutions of the one-dimensional Heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ by the method of separation of variables.

Consider $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Assume that that the solution of heat equation is of the form $u = X(x)T(t)$ where X is a function of x and T is a function of t only.

Hence the PDE becomes

$\frac{d(XT)}{dt} = c^2 \frac{d^2(XT)}{dx^2}$ OR $X\frac{dT}{dt} = c^2 T \frac{d^2X}{dx^2}$ and dividing by XT we have

$\frac{1}{c^2 T}\frac{dT}{dt} = \frac{1}{x}\frac{d^2X}{dx^2}$

Equating both sides to a common constant k we have,

$\frac{1}{c^2 T}\frac{dT}{dt} = k$ and $\frac{1}{x}\frac{d^2X}{dx^2} = k$

$\frac{dT}{dt} - c^2 kT = 0$ and $\frac{d^2X}{dx^2} - kx = 0$

Or $(D - c^2 k)T = 0$ and $(D - k)X = 0$

Where $D = \frac{d}{dt}$ in the first equation and $D = \frac{d}{dx}$ in the second equation

Case (i) : Let k = 0

Auxiliary equations are m=0 and m=0,m=0 are the roots.

Solutions are given by

$T = c_1 e^{0t} = c_1$ and $X = (c_2 x + c_3)$

Hence the solution of the PDE is given by

$u = c_1 (c_2 x + c_3)$

Case (ii) : Let k = p^2

Auxiliary equations are $m - c^2 p^2 = 0$ and $m^2 - p^2 = 0$

$m = c^2 p^2$ and $m = \pm p$

Solutions are given by

$T = c'_1 e^{c^2 p^2 t}$ and $X = (c'_2 e^{px} + c'_3 e^{-px})$

Hence the solution of the PDE is given by

$u = c'_1 e^{c^2 p^2 t} (c'_2 e^{px} + c'_3 e^{-px})$

case (iii): Let $k = -p^2$

Auxiliary equations are $m + c^2p^2 = 0$ and $m^2 + p^2 = 0$

$m = -c^2p^2$ and $m = \pm ip$

Solutions are given by

$T = c''_1\, e^{-c^2p^2t}$ and $X = (c''_2 \cos px + c''_3 \sin px)$

Hence the solution of the PDE is given by

$u = c''_1\, e^{-c^2p^2t}\,(c''_2 \cos px + c''_3 \sin px)$

Of these three solutions, we have to choose that the solution which is consistent with the physical nature of the problem. As we will be dealing with problem of vibrations, u must be a periodic function of x and t. Hence their solution must involve trigonometric terms. Accordingly the solution given by case(iii) i.e., of the form

$u = c''_1\, e^{-c^2p^2t}\,(c''_2 \cos px + c''_3 \sin px)$ is the only suitable solution of the Heat equation.