NAGARJUNA COLLEGE OF ENGINEERING AND TECHNOLOGY

(An autonomous institution under VTU)

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NAGARJUNA
COLLEGE OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

Advanced Calculus And Numerical Methods

(COURSE CODE 22MATS21/22MATC21/22MATE21)

CLASS NOTES FOR II SEM B.E.

Module-4

DIFFERENTIAL EQUATIONS OF HIGHER ORDER

DIFFERENTIAL EQUATIONS OF HIGHER ORDER

Higher-order linear ODE's with constant coefficients - Inverse differential operator(Particular integral for (e^{ax} , $sin\ ax, cos\ ax, x^m$ only), method of variationof parameters, Cauchy's and Legendre homogeneous differential equations. Problems

Linear differential equation with constant coefficients:

A differential equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots\ldots\ldots + a_{n-1} \frac{dy}{dx} + a_n y = \phi(x) \ldots\ldots\ldots (1).$$

is called linear differential equations of order n with constant coefficient. Where

$a_0, a_1, a_2, \ldots\ldots, a_n,$ are constants and $\phi(x)$ is a function of x.

Solution of second and higher order linear differential equations:

Equation (1) can be put in the form of

$$[a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots\ldots + a_{n-1} D + a_n] y = \phi(x) \ldots\ldots\ldots (2).$$

Where $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, $D^3 = \frac{d^3}{dx^3}$, ………… and D is called the differential operator.

Equation (2) is of the form $f(D)y = \phi(x) \ldots\ldots\ldots (3)$.

Where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots\ldots + a_{n-1} D + a_n$, is a polynomial in D of degree n.

The solution of (2) contains two parts. The first part which gives zero when operated by f(D) is called complementary function (C.F.) and second part which gives $\phi(x)$ when operated by f(D) is called the particular integral (P.I.). Therefore, complete solution of (1) is $y = C.F. + P.I.$ i.e., $y = y_c + y_p$.

Note:

If $\phi(x) = 0$ in (1), then the equation is called a homogeneous linear differential equation (L.D.E) and its solution is $y = C.F.$ i.e., $y = y_c$.

Rules for finding the C.F:

We first write the auxiliary equation of (1) as $f(D) = 0$ or $f(m) = 0$.

i.e. $a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots\ldots + a_{n-1} D + a_n = 0$. Depend on the nature of the roots of auxiliary equation $f(m) = 0$, we write the C.F. as follows.

1. If the roots are real and distinct say m_1, m_2, m_3……., then
   y_c = C.F = $c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$, where $c_1, c_2, c_3, \ldots, c_n$ are constants.

2. (i) If two roots are equal say $m_1 = m_2 = m$ and other are real and distinct, then
   y_c = C.F = $(c_1 + c_2 x)\, e^{mx} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots\ldots + c_n e^{m_n x}$.

   (ii) If three roots are equal say $m_1 = m_2 = m_3 = m$ and other are real and distinct, then
   y_c = C.F = $(c_1 + c_2 x + c_3 x^2)\, e^{mx} + c_4 e^{m_4 x} + c_5 e^{m_5 x} + \ldots\ldots + c_n e^{m_n x}$ and so on.

3. (i) If there is a pair of complex roots $m_1, m_2 = a \pm ib$ and others are real and distinct, then C.F = $e^{ax}(c_1 \cos bx + c_2 \sin bx) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots\ldots$

   (ii) If there is a pair of repeated complex roots say $m_1, m_2 = a \pm ib$ and $m_3, m_4 = a \pm ib$ and others are real and distinct. then,
   C.F = $e^{ax}[(c_1 + c_2 x)\cos bx + (c_3 + c_4 x)\sin bx] + c_5 e^{m_5 x} + \ldots\ldots$

Solutions of the Homogeneous linear differential equations:

1. Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0.$

Solution:

Given $(D^2 + 5D + 6)\, y = 0$……….(1). Where $D = \frac{d}{dx}$ and $D^2 = \frac{d^2}{dx^2}$.

Equation (1) is of the form f(D) = 0.

∴ The auxiliary equation (A.E.) of (1) is $m^2 + 5m + 6 = 0$. By factorisation, we get,

$m^2 + 3m + 2m + 6 = 0$. ∴ $m(m + 3) + 2(m + 3) = 0$. ∴ $(m + 3)(m + 2) = 0$.

∴ $(m + 3) = 0$ and $(m + 2) = 0$. ∴ $m = -3$ and $m = -2$.

∴ The roots $m = -2, -3$ are real and distinct. ∴ $\mathrm{C.F} = C_1 e^{-2x} + C_2 e^{-3x}$.

∴ The solution is y = C.F. i.e., $y = C_1 e^{-2x} + C_2 e^{-3x}$.

2. Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0.$

Solution:

Given $(D^3 - 3D^2 + 3D - 1)\, y = 0$ ……… (1). Where $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, and $D^3 = \frac{d^3}{dx^3}$.

Equation (1) is of the form f(D) = 0.

∴ The auxiliary equation of (1) is $m^3 - 3m^2 + 3m - 1 = 0$.

∴ $(m - 1)^3 = 0$. Using $a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3$

∴ The roots $m = 1, 1, 1$. Roots are real and repeated.

$\therefore\ C.F = (C_1 + C_2x + C_3 x^2)\, e^x.$

$\therefore$ The solution is $y = C.F.$ i.e., $y = (C_1 + C_2x + C_3 x^2)\, e^x.$

3. Solve $\frac{d^3y}{dx^3} + y = 0.$

Solution:

Given $(D^3 + 1)\, y = 0$ ……… (1). Where $D^3 = \frac{d^3}{dx^3}$.

Equation (1) is of the form $f(D) = 0$.

$\therefore$ The auxiliary equation of (1) is $m^3 + 1 = 0$.

By inspection method, $m = -1$ is a root.

By synthetic division method, we get,

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & 0 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

$\therefore\ m^2 - m + 1 = 0.$

$\therefore\ m = \frac{-(-1)\pm\sqrt{(-1)^2-4\times1\times1}}{2\times1} = \frac{1\pm\sqrt{1-4}}{2} = \frac{1\pm\sqrt{-3}}{2} = \frac{1\pm i\sqrt{3}}{2} = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$

$\therefore$ The roots $m_1 = -1$(real), and $m_2, m_3 = \frac{1}{2} \pm \frac{i\sqrt{3}}{2} = a \pm ib$. (a pair of complex roots).

$\therefore\ C.F = C_1e^{-x} + e^{\frac{1}{2}x}\left(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right).$

$\therefore$ The solution is $y = C.F.$ i.e., $y = C_1e^{-x} + e^{\frac{1}{2}x}\left(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right).$

4. Solve $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0.$

Solution:

Given $(D^4 + 8D^2 + 16)\, y = 0$ ……… (1). Where $D^2 = \frac{d^2}{dx^2}$, and $D^4 = \frac{d^4}{dx^4}$.

Equation (1) is of the form $f(D) = 0$.

$\therefore$ The auxiliary equation of (1) is $m^4 + 8\, m^2 + 16 = 0.$

$\therefore\ (m^2 + 4)^2 = 0.\ \therefore\ m^2 + 4 = 0.\ \therefore\ m^2 = -4.\ \therefore\ m = \pm\sqrt{-4} = \pm\, i\sqrt{4} = \pm\, i\, 2.$

$\therefore$ The roots $m_1, m_2 = 0 \pm i\, 2$, and $m_3, m_4 = 0 \pm i\, 2$, a pair of repeated complex roots.

$\therefore\ C.F = e^{0\,x}\,[(C_1 + C_2x)\cos 2x + (C_3 + C_4x)\sin 2x].$

i.e., $C.F = (C_1 + C_2x)\cos 2x + (C_3 + C_4x)\sin 2x.$

The solution is $y = C.F.$ i.e, $y = (C_1 + C_2x)\cos 2x + (C_3 + C_4x)\sin 2x$.

5. Solve $(4D^4 - 8D^3 - 7D^2 + 11D + 6)\,y = 0$.

Solution:

Given $(4D^4 - 8D^3 - 7D^2 + 11D + 6)\,y = 0$ ……… (1). Where $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$ …….

Equation (1) is of the form $f(D) = 0$.

∴ The auxiliary equation of (1) is $4m^4 - 8m^3 - 7m^2 + 11m + 6 = 0$.

By Inspection method, $m = -1$ is a root.

By synthetic division method, we get,

−1	4	−8	−7	11	6
	0	−4	12	−5	−6
	4	−12	5	6	0

∴ $4m^3 - 12m^2 + 5m + 6 = 0$.

By Inspection method, $m = 2$ is another root.

Again, by synthetic division method, we get,

2	4	−12	5	6
	0	8	−8	−6
	4	−4	−3	0

∴ $4m^2 - 4m - 3 = 0$.

∴ $4m^2 - 6m + 2m - 3 = 0$.

∴ $2m(2m-3) + 1(2m-3) = 0$.

∴ $(2m-3)(2m+1) = 0$. ∴ $(2m-3) = 0$ and $(2m+1) = 0$.

∴ $m = \frac{3}{2}$ and $m = \frac{-1}{2}$.

∴ The roots are $m_1 = -1$, $m_2 = 2$, $m_3 = \frac{3}{2}$, and $m_4 = \frac{-1}{2}$.

These roots are real and distinct.

∴ $C.F = C_1e^{-x} + C_2e^{2x} + C_3e^{\frac{3}{2}x} + C_4\,e^{-\frac{1}{2}x}$.

∴ The solution is $y = C.F.$ i.e., $y = C_1e^{-x} + C_2e^{2x} + C_3e^{\frac{3}{2}x} + C_4\,e^{-\frac{1}{2}x}$.

6. Solve $\frac{d^4x}{dt^4} + 4x = 0$.

Solution:

Given $(D^4 + 4)\,x = 0$ ……… (1). Where $D^4 = \frac{d^4}{dt^4}$.

Equation (1) is of the form $f(D) = 0$.

∴ The auxiliary equation of (1) is $m^4 + 4 = 0$. ∴ $(m^2)^2 + 2^2 = 0$.

Using $a^2 + b^2 = (a + b)^2 - 2ab = 0$, we get,

$(m^2 + 2) - 2m = 0$ and $(m^2 + 2) + 2m = 0$.

i.e., $m^2 - 2m + 2 = 0$ and $m^2 + 2m + 2 = 0$.

$\therefore \quad m = \frac{-(-2)\pm\sqrt{(-2)^2-4\times1\times2}}{2\times1} = \frac{2\pm\sqrt{-4}}{2} = \frac{2\pm2i}{2} = 1 \pm i$, and

$m = \frac{-2\pm\sqrt{2^2-4\times1\times2}}{2\times1} = \frac{-2\pm\sqrt{-4}}{2} = \frac{-2\pm2i}{2} = -1 \pm i$.

$\therefore$ The roots $m_1, m_2 = 1 \pm i$, and $m_3, m_4 = - \pm i\, 2$, two pairs of complex roots.

$\therefore \quad C.F = e^t (C_1 \cos t + C_2 \sin t) + e^{-t} (C_3 \cos t + C_4 \sin t)$.

$\therefore$ The solution is y = C. F. i.e., $y = e^t (C_1 \cos t + C_2 \sin t) + e^{-t} (C_3 \cos t + C_4 \sin t)$.

HOME WORK:

1. Solve $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$.

2. Solve $(D^3 + D^2 + 4D + 4)\, y = 0$.

3. Solve $(D^2 + 1)^3 y = 0$.

4. Solve $\frac{d^4x}{dt^4} - 2\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} = 0$.

5. Solve $4y''' + 4y'' + y' = 0$.

6. Solve $(D^4 + 2D^3 - 5D^2 - 6D)\, y = 0$.

7. Solve $(D^5 - D^4 - D + 1)\, y = 0$.

8. Solve $(D^4 - 4D^3 - 5D^2 - 36D - 36)\, y = 0$.

9. Solve $4\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} - 23\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36\, y = 0$.

Ans : 1. $y = (C_1 + C_2 t)\, e^{-3t}$.

2. $y = C_1 e^{-x} + C_2 \cos 2x + C_3 \sin 2x$.

3. $y = (C_1 + C_2 x + C_3 x^2) \cos x + (C_4 + C_5 x + C_6 x^2) \sin x$

4. $y = C_1 + C_2 t + (C_3 + C_4 t)\, e^t$.

5. $y = C_1 + (C_2 + C_3 x)\, e^{-\frac{1}{2}x}$.

6. $y = C_1 + C_2 e^{-x} + C_3 e^{-3x} + C_4 e^{2x}$.

7. $y = C_1 e^{-x} + (C_2 + C_3 x)\, e^x + (C_4 \cos x + C_5 \sin x)$.

8. $y = C_1 e^{-x} + C_2 e^{6x} + e^{-\frac{1}{2}x} (C_3 \cos\frac{\sqrt{23}}{2}x + C_4 \sin\frac{\sqrt{23}}{2}x)$.

9. $y = (C_1 + C_2 x)\, e^{2x} + (C_3 + C_4 x) e^{-3\backslash 2\, x}$

Inverse Differential operator method:

As D is called the differential operator, $\frac{1}{D}$ is called the inverse differential operator.

Where D stands for differentiation and $\frac{1}{D}$ stands for integration.

Rules for finding the particular integral by using inverse differential operator method:

We first write the particular integral as P.I. $= \frac{1}{f(D)}\phi(x)$. Depend on the form of the function $\phi(x)$ we write the P.I as follows.

1. If $\phi(x) = e^{ax}$, then replace D by a in f(D).

   (i) P.I. $= \frac{1}{f(a)} e^{ax}$, where $f(a) \neq 0$. [a is not a root of the A.E. $f(m) = 0$].

   (ii) P.I. $= x \frac{1}{f^1(a)} e^{ax}$, where $f(a) = 0$ and $f'(a) \neq 0$. [a is a root of the A.E. $f(m) = 0$].

   (iii) P.I. $= x^2 \frac{1}{f''(a)} e^{ax}$, where $f(a)=0$, $f'(a)=0$ and $f''(a) \neq 0$. [a is double root of the A.E],

   and so on.

2. If $\phi(x) = \sin(ax+b)$ or $\cos(ax+b)$, then rewrite f(D) in the form of $F(D^2)$ and replace D^2 by $-a^2$.

   (i) P.I. $= \frac{1}{F(-a^2)}$ [sin (a x + b) or cos(a x + b)], where $F(-a^2) \neq 0$.

   (ii) P.I. $= x \frac{1}{F'(-a^2))}$ sin (a x + b) or cos (a x + b), where $F(-a^2) = 0$ and $F'(-a^2) \neq 0$.

   (iii) P.I. $= x^2 \frac{1}{F''(-a^2)}$ sin (a x + b) or cos(a x + b), where $F(-a^2)=0$, $f'(-a^2)=0$ and

   $F''(-a^2) \neq 0$.

3. If $\phi(x) = x^m$, m being a + ve integer or $\phi(x) = a_0x^m + a_1x^{m-1} + \ldots\ldots. + a_m$, a polynomial in x, then write $\phi(x)$ in descending powers of x and f(D) in ascending powers of D and then divide $\phi(x)$ by f(D) till to obtain the remainder as zero. The resulting quotient is the P.I.

Problems:

1. Solve $(D^2 + 5D + 6)y = e^x$.

Solution:

Given $(D^2 + 5D + 6)y = e^x$ ....(1).   Where $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$ ....

Equation (1) is of the form $f(D) = \phi(x)$.

$\therefore$ The auxiliary equation of (1) is $m^2 + 5m + 6 = 0$. By factorisation, we get,

$m^2 + 3m + 2m + 6 = 0. \quad \therefore \; m(m + 3) + 2(m + 3) = 0. \quad \therefore \; (m + 3)(m + 2) = 0.$

$\therefore \; (m + 3) = 0$ and $(m + 2) = 0. \quad \therefore \; m = -3$ and $m = -2.$

$\therefore$ The roots $m_1 = -2, \; m_2 = -3$ are real and distinct. $\therefore$ C.F. $= C_1 e^{-2x} + C_2 e^{-3x}$.

Now, P.I. $= \frac{1}{f(D)} \phi(x) = \frac{1}{D^2 + 5D + 6} e^x$. Replace $D = a = 1$. [Using rule 1(i)]

$\therefore$ P.I. $= \frac{e^x}{1^2 + 5 \times 1 + 6} = \frac{e^x}{12}$.

$\therefore$ The complete solution of (1) is $y = C.F + P.I.$

$\therefore \; y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{e^x}{12}$.

2. Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$.

Solution:

Given $(D^2 - 6D + 9)y = 6e^{3x} + 7e^{-2x} - \log 2$.........(1). Where $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$ ....

Equation (1) is of the form $f(D) = \phi(x)$.

$\therefore$ The auxiliary equation of (1) is $m^2 - 6m + 9 = 0. \quad \therefore \; (m - 3)^2 = 0.$

$\therefore \; (m - 3)(m - 3) = 0.$

$\therefore \; (m - 3) = 0$ and $(m - 3) = 0. \quad \therefore \; m = 3$ and $m = 3$

$\therefore$ The roots $m_1 = 3$ and $m_2 = 3$ are real and repeated. $\therefore$ C.F. $= (C_1 + C_2 x) e^{3x}$.

Now, P.I. $= \frac{1}{f(D)} \phi(x) = \frac{1}{D^2 - 6D + 9}(6e^{3x} + 7e^{-2x} - \log 2)$.

$\therefore$ P.I. $= \frac{6e^{3x}}{D^2 - 6D + 9} + \frac{7e^{-2x}}{D^2 - 6D + 9} + \frac{-\log 2}{D^2 - 6D + 9} = P_1 + P_2 + P_3$.

Now, $P_1 = \frac{6e^{3x}}{D^2 - 6D + 9}$. Since 3 is a double root of the A.E., using rule 1(iii), we get,

$P_1 = \frac{6x^2 e^{3x}}{2}$. $\quad [\because f''(D) = 2]$

$P_2 = \frac{7e^{-2x}}{D^2 - 6D + 9}$. Since -2 is not a root of the A.E., using rule 1(i), we get,

$P_2 = \frac{7e^{-2x}}{(-2)^2 - 6(-2) + 9} = \frac{7e^{-2x}}{25}$.

$P_3 = \frac{-\log 2}{D^2 - 6D + 9} = \frac{-(\log 2)e^{0x}}{D^2 - 6D + 9} = \frac{-\log 2 \, e^{0x}}{0^2 - 6 \times 0 + 9} = \frac{-\log 2}{9}$. [Put $D = 0$, using rule 1(i)].

$\therefore$ P.I. $= P_1 + P_2 + P_3 = \frac{6x^2 e^{3x}}{2} + \frac{7e^{-2x}}{25} - \frac{\log 2}{9}$.

$\therefore$ The solution is $y = C.F + P.I.$

$\therefore \quad y = (C_1 + C_2 x)\, e^{3x} + \frac{6\, x^2\, e^{3x}}{2} + \frac{7e^{-2x}}{25} - \frac{\log 2}{9}$.

3. Solve $(D^3+1)y = \cos(2x-1)$.

Solution:

Given $(D^3 + 1)\, y = \cos(2x-1)$……… (1). Where $\frac{d^3}{dx^3} = D^3$.

Equation (1) is of the form $f(D) = \phi(x)$.

$\therefore$ The auxiliary equation of (1) is $m^3 + 1 = 0$. By inspection method $m = -1$ is the root.

By synthetic division method, we get,

−1	1	0	0	1
	0	−1	1	−1
	1	−1	1	0

$\therefore \quad m^2 - m + 1 = 0.$

$\therefore \quad m = \frac{-(-1)\pm\sqrt{(-1)^2-4\times1\times1}}{2\times1} = \frac{1\pm\sqrt{1-4}}{2} = \frac{1\pm\sqrt{-3}}{2} = \frac{1\pm i\sqrt{3}}{2} = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$

$\therefore$ The roots are $m_1 = -1$(real) and $m_2, m_3 = \frac{1}{2} \pm \frac{i\sqrt{3}}{2} = a \pm ib$. (a pair of complex roots).

$\therefore \quad C.F = C_1 e^{-x} + e^{\frac{1}{2}x}\left(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right).$

Now, $P.I. = \frac{1}{f(D)}\, \phi(x) = \frac{\cos(2x-1)}{D^3+1} = \frac{\cos(2x-1)}{D.D^2+1}$. [Using rule 2(i), put $D^2 = -a^2 \rightarrow -2^2 = -4$].

$\therefore \quad P.I. = \frac{\cos(2x-1)}{D\times(-4)+1} = \frac{1}{(-4D+1)} \times \frac{(1+4D)}{(1+4D)} = \frac{(1+4D)}{(1-4D)(1+4D)} \cos(2x-1)$

$\therefore \quad P.I. = \frac{\cos(2x-1)+4\,D\cos(2x-1)}{1-16D^2}$ [*Put* $D^2 = -2^2 = -4$]

$\therefore \quad P.I. = \frac{\cos(2x-1) - 8\sin(2x-1)}{1-16(-4)} = \frac{\cos(2x-1) - 8\sin(2x-1)}{65}$. [$\because D\cos(2x-1) = -2\sin(2x-1)$]

$\therefore$ The solution is $y = C.F + P.I.$

$\therefore \quad y = C_1 e^{-x} + e^{\frac{1}{2}x}\left(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)]$

4. Solve $(D^2 - 4D + 3)\, y = \sin 3x \cos 2x$.

Solution:

Given $(D^2 - 4D + 3)y = \sin 3x\ \cos 2x$ …… (1). Where $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$.

Equation (1) is of the form $f(D) = \phi(x)$.

$\therefore$ The auxiliary equation of (1) is $m^2 - 4m + 3 = 0$. $\therefore$ $(m-1)(m-3) = 0$.

$\therefore\ (m-1)=0$ and $(m-3)=0.\quad \therefore\ m=1$ and $m=3.$

$\therefore$ The roots $m_1=1$ and $m_2=3$ are real and distinct. $\therefore$ C. F. $=C_1\,e^{x}+C_2\,e^{3x}.$

Now, P. I. $=\frac{1}{f(D)}\,\phi(x)=\frac{1}{D^2-4D+3}\sin 3x\ \cos 2x=\frac{1}{(D^2-4D+3)}\frac{1}{2}(\sin 5x+\sin x).$

$\therefore$ P. I.$=\frac{1}{2}\left[\frac{\sin 5x}{D^2-4D+3}\right]+\frac{1}{2}\left[\frac{\sin x}{D^2-4D+3}\right]=P_1+P_2.$

Now, $P_1=\frac{1}{2}\left[\frac{\sin 5x}{D^2-4D+3}\right]$ [Using rule 2, put $D^2=-a^2=-5^2=-25$]

$$=\frac{1}{2}\left[\frac{\sin 5x}{-25-4D+3}\right]=\frac{1}{2}\left[\frac{\sin 5x}{-22-4D}\right]=-\frac{1}{4}\left[\frac{1}{(2D+11)}\times\frac{(2D-11)}{(2D-11)}\sin 5x\right]$$

$$=-\frac{1}{4}\left[\frac{2D(\sin 5x)-11\sin 5x}{4D^2-121}\right]=-\frac{1}{4}\left[\frac{10(\cos 5x)-11\sin 5x}{4\times(-25)-121}\right]$$ By putting $D^2=-5^2=-25.$

$\therefore\ P_1=-\frac{1}{4}\left[\frac{10(\cos 5x)-11\sin 5x}{-221}\right]=\left[\frac{10(\cos 5x)-11\sin 5x}{884}\right].$

And $P_2=\frac{1}{2}\left[\frac{\sin x}{D^2-4D+3}\right]$ [*Using rule* 2, *put* $D^2=-a^2=-1$].

$$=\frac{1}{2}\left[\frac{\sin x}{-1-4D+3}\right]=\frac{1}{2}\left[\frac{\sin x}{-4D+2}\right]=\frac{1}{4}\left[\frac{1}{(1-2D)}\times\frac{(1+2D)}{(1+2D)}\right]\sin x=\frac{1}{4}\left[\frac{\sin x+2D\sin x}{1-4D^2}\right]$$

$\therefore\ P_2=\frac{1}{4}\left[\frac{\sin x+2\cos x}{1-4(-1)}\right]=\frac{1}{4}\left[\frac{\sin x+2\cos x}{5}\right]=\left[\frac{\sin x+2\cos x}{20}\right]$

$\therefore$ P. I. $=P_1+P_2=\left[\frac{10(\cos 5x)-11\sin 5x}{884}\right]+\left[\frac{\sin x+2\cos x}{20}\right]$

The complete solution is $y=$ C. F $+$ P. I.

$\therefore\ y=C_1\,e^{3x}+C_2\,e^{x}+\left[\frac{10(\cos 5x)-11\sin 5x}{884}\right]+\left[\frac{\sin x+2\cos x}{20}\right].$

5. Solve $(D^2+2D+1)y=x^2+2x.$

Solution:

Given $(D^2+2D+1)y=x^2+2x$ .... (1). Where $D=\frac{d}{dx},\ D^2=\frac{d^2}{dx^2}.$

Equation (1) is of the form $f(D)=\phi(x).$

$\therefore$ The auxiliary equation of (1) is $m^2+2m+1=0.\quad \therefore\ (m+1)^2=0.$

$\therefore\ (m+1)=0$ and $(m+1)=0.\quad \therefore\ m=-1$ and $m=-1.$

$\therefore$ The roots $m_1=-1$ and $m_2=-1$ are real and repeated. $\therefore$ C. F $=(C_1+C_2x)e^{-x}.$

Now, P. I. $=\frac{1}{f(D)}\,\phi(x)=\frac{x^2+2x+4}{D2+2D+1}=\frac{x^2+2x+4}{1+2D+D^2}$. [Use rule 3].

$$
\begin{array}{r|l}
 & x^2 - 2x + 2 \\
\hline
1+2D+D^2 & x^2 + 2x \\
 & x^2 + 4x + 2 \\
\hline
\text{Sub.} & 0 - 2x - 2 \\
 & \quad - 2x - 4 \\
\hline
\text{Sub.} & \qquad 0 + 2 \\
 & \qquad\quad 2 \\
\hline
\text{Sub.} & \qquad\quad 0 \\
\hline
\end{array}
$$

$\therefore$ $P.I = x^2 - 2x + 2$. $\therefore$ The complete solution is $y = C.F + P.I.$

$\therefore$ $y = (C_1 + C_2)e^{-x} + x^2 - 2x + 2.$

6. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4.$

Solution:

Given $(D^2 + D)y = x^2 + 2x + 4$.......... (1). Where $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$.

Equation (1) is of the form $f(D) = \phi(x)$.

$\therefore$ The auxiliary equation of (1) is $m^2 + m = 0$. $\therefore$ $m(m+1) = 0$.

$\therefore$ $m = 0$ and $(m+1) = 0$. $\therefore$ $m = 0$ and $m = -1$.

$\therefore$ The roots $m_1 = 0$ and $m_2 = -1$ are real and distinct. $\therefore$ $C.F = C_1 e^{0x} + C_2 e^{-x}$.

$\therefore$ $C.F = C_1 + C_2 e^{-x}$.

Now, $P.I. = \frac{1}{f(D)} \phi(x) = \frac{x^2+2x+4}{D^2+D} = \frac{x^2+2x+4}{D+D^2}$. [Use rule 3].

$$
\begin{array}{r|l}
 & \frac{x^3}{3} + 4x \\
\hline
D+D^2 & x^2 + 2x + 4 \\
 & x^2 + 2x \\
\hline
\text{Sub.} & 0 + 0 + 4 \\
 & \qquad\quad 4 \\
\hline
\text{Sub.} & \qquad\quad 0 \\
\hline
\end{array}
$$

$\therefore$ $P.I = \frac{x^3}{3} + 4x$

$\therefore$ The complete solution is $y = C.F + P.I.$

$\therefore$ $y = C_1 + C_2 e^{-x} + \frac{x^3}{3} + 4x.$

7. Solve $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x.$

Solution :

Given $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$. Where $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$.

$\therefore$ $(D+2)(D-1)^2 y = e^{-2x} + e^x - e^{-x}$.......(1). [$\because$ $2\sinh x = e^x - e^{-x}$]

Equation (1) is of the form $f(D) = \phi(x)$.

$\therefore$ The auxiliary equation of (1) is $(m+2)(m-1)^2 = 0$.

$\therefore$ $(m+2)=0$ and $(m-1)^2 = 0$. $\therefore$ $m = -2$ and $m = 1, 1$

$\therefore$ The roots are $m_1 = -2$ (real), $m_2, m_3 = 1$. (Real and repeated).

$\therefore$ $C.F. = C_1 e^{-2x} + (C_2 + C_3 x) e^x$.

Now, $P.I. = \frac{1}{(D+2)(D-1)^2}[e^{-2x} + e^x - e^{-x}]$.

$\therefore$ $P.I. = \frac{1}{(D+2)(D-1)^2} e^{-2x} + \frac{1}{(D+2)(D-1)^2} e^x - \frac{1}{(D+2)(D-1)^2} e^{-x} = P_1 + P_2 - P_3$.

$\therefore$ $P_1 = \frac{1}{(D+2)(D-1)2} e^{-2x}$. [Using rule 1(ii), as -2 is a root of the A.E.].

$$= \frac{x}{(D+2)\,2(D-1)+(D-1)^2} e^{-2x} = \frac{x\,e^{-2x}}{(-2+2)2(-2-1)+(-2-1)^2} = \frac{x\,e^{-2x}}{0+9} = \frac{x\,e^{-2x}}{9}.$$

$P_2 = \frac{e^x}{(D+2)(D-1)^2}$ [Using rule 1(iii), as 1 is a double root of the A.E.]

$$= \frac{x\,e^x}{(D+2)2(D-1)+(D-1)^2} = \frac{x^2 e^x}{2\{(D+2)+(D-1)\}+2(D-1)} = \frac{x^2 e^x}{2\{(1+2)+(1-1)\}+2(1-1)} = \frac{x^2 e^x}{6}.$$

$P_3 = \frac{e^{-x}}{(D+2)(D-1)^2}$ [Using rule 1(i), put $D = a = -1$]

$$= \frac{e^{-x}}{(-1+2)(-1-1)^2} = \frac{e^{-x}}{4}.$$

$\therefore$ $P.I. = \frac{x\,e^{-2x}}{9} + \frac{x^2 e^x}{6} - \frac{e^{-x}}{4}$.

$\therefore$ The solution is $y = C.F + P.I.$

$\therefore$ $y = C_1 e^{-2x} + (C_2 + C_3 x)\, e^x + \frac{x\,e^{-2x}}{9} + \frac{x^2 e^x}{6} - \frac{e^{-x}}{4}$.

8. *Solve* $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$.

Solution:

Given $(D^3 + 4D)y = \sin 2x$ .... (1). Where $D = \frac{d}{dx}$, $D^3 = \frac{d^3}{dx^3}$.

Equation (1) is of the form $f(D) = \phi(x)$.

$\therefore$ The auxiliary equation of (1) is $m^3 + 4m = 0$. $\therefore$ $m(m^2 + 4) = 0$.

$\therefore$ $m = 0$ and $(m^2 + 4) = 0$. $\therefore$ $m = 0$ and $m = \pm 2i$.

$\therefore$ The roots $m_1 = 0$ (real) and $m_2, m_3 = \pm 2i = 0 \pm 2i$ (a pair of complex roots).

$\therefore \quad C.F = C_1 e^{0x} + e^{0x}(C_2 \cos 2x + C_3 \sin 2x)$.

$\therefore \quad C.F = C_1 + (C_2 \cos 2x + C_3 \sin 2x)$.

Now, $P.I. = \frac{1}{f(D)} \phi(x) = \frac{1}{D^3+4D}(\sin 2x) = \frac{\sin 2x}{D^2.D+4D}$. [Using rule 2, put $D^2 = -a^2 = -2^2$].

$= \frac{\sin 2x}{-2^2.D+4D} = \frac{\sin 2x}{-4D+4D}$. But denominator is zero. [Use rule 2(ii)]

$\therefore \quad P.I. = \frac{\sin 2x}{D^3+4D} = \frac{x.\sin 2x}{3D^2+4} = \frac{x.\sin 2x}{3(-2^2)+4} = \frac{x.\sin 2x}{-8} = -\frac{x.\sin 2x}{8}$.

$\therefore$ The solution is $y = C.F + P.I = C_1 + C_2 \cos 2x + C_3 \sin 2x - \frac{x.\sin 2x}{8}$.

9. Solve $y'' - 2y' + 2y = x$

Solution:

Given $(D^2 - 2D + 2)y = x$ .... (1). Where $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$.

Equation (1) is of the form $f(D) = \phi(x)$.

$\therefore$ The auxiliary equation of (1) is $m^2 - 2m + 2 = 0$.

$\therefore \quad m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4\times1\times2}}{2\times1} = \frac{2\pm\sqrt{-4}}{2} = \frac{2\pm2i}{2} = 1 \pm i,$

$\therefore$ The roots $m_1, m_2 = 1 \pm i$, a pairs of complex roots.

$\therefore \quad C.F = e^x(C_1 \cos x + C_2 \sin x)$.

$\therefore P.I. = \frac{1}{f(D)} \phi(x) = \frac{1}{D^2-2D+2}(x + e^x \cos x) = \frac{x}{D^2-2D+2}$

Now, $P.I = \frac{x}{D^2-2D+2}$. (Use rule 3.)

$$
\begin{array}{r|l}
 & \frac{x}{2} + \frac{1}{2} \\
\hline
2 - 2D + D^2 & x \\
 & x - 1 \\
\hline
\text{Sub.} & \quad 1 \\
 & \quad 1 \\
\hline
\text{Sub.} & \quad 0 \\
\end{array}
$$

$\therefore P.I = \frac{x}{2} + \frac{1}{2}$.

$\therefore$ The solution is $y = C.F + P.I. = e^x\{C_1 \cos x + C_2 \sin x\} + \frac{x}{2} + \frac{1}{2}$.

10. Solve $(D^3 + 2D^2 + D)\, y = \sin^2 x$.

Solution :

Given $(D^3+2D^2+D)y = \sin^2 x$............(1).

The Auxiliary equation is $m^3+2m^2+m=0$. $\therefore$ $m(m^2+2m+1)=0$.

$\therefore$ $m(m+1)^2=0$. $\therefore$ $m(m+1)(m+1)=0$.

$\therefore$ $m=0,\ m=-1,\ m=-1$ are the roots.

$\therefore$ $C.F = C_1+(C_2+C_3x)e^{-x}$

Now, $\text{P.I.} = \frac{1}{D^3+2D^2+D}(\sin^2 x)$

$= \frac{1}{D^3+2D^2+D}\left(\frac{1-\cos 2x}{2}\right).$

$= \frac{1}{2}.\frac{1}{D^3+2D^2+D}e^{0x} - \frac{1}{2}.\frac{1}{D^3+2D^2+D}\cos 2x.$

$= \frac{1}{2}.\frac{1}{D^3+2D^2+D}e^{0x} - \frac{1}{D^3+2D^2+D}\cos 2x = P_1 - P_2.$

$P_1 = \frac{1}{2}.\frac{1}{D^3+2D^2+D}e^{0x} = \frac{1}{2}.\frac{1}{3D^2+4D+1}e^{0x} = \frac{x}{2}.\frac{1}{0+0+1}e^{0x} = \frac{x}{2}.$

$P_2 = \frac{1}{2}.\frac{1}{D^3+2D^2+D}\cos 2x = \frac{1}{2}.\frac{1}{-4D-8+D}\cos 2x = \frac{1}{2}.\frac{1}{-3D-8}\cos 2x = -\frac{1}{2}.\frac{1}{(3D+8)}\frac{(3D-8)}{(3D-8)}\cos 2x.$

$\therefore$ $P_2 = -\frac{1}{2}\frac{(3D-8)}{(9D^2-64)}\cos 2x = -\frac{1}{2}\frac{(3D-8)}{(-36-64)}\cos 2x = \frac{1}{200}[3D\cos 2x - 8\cos 2x].$

$\therefore$ $P_2 = \frac{1}{200}[-6\sin 2x - 8\cos 2x] = \frac{-1}{200}[6\sin 2x + 8\cos 2x].$

$\therefore$ $\text{P.I.} = \frac{x}{2} + \frac{1}{200}[6\sin 2x + 8\cos 2x].$

$\therefore$ $\text{P.I.} = \frac{x}{2} + \frac{1}{100}[3\sin 2x + 4\cos 2x].$

$\therefore$ *The solution is* $y = C.F.+P.I.$

$\therefore$ $y = C_1+(C_2+C_3x)e^{-x} + \frac{x}{2} + \frac{1}{100}[3\sin 2x + 4\cos 2x].$

11. Solve $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=(1-e^x)^2$.

Solution:

Given $(D^2+D+1)y = (1-e^x)^2$..........(1).

The Auxiliary equation is $m^2+m+1=0$.

$\therefore$ $m = \frac{-1\pm\sqrt{(1)^2-4\times1\times1}}{2\times1} = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm\sqrt{-3}}{2} = \frac{-1\pm i\sqrt{3}}{2} = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$

$\therefore$ The roots are $m_1, m_2 = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2} = a \pm ib$. (a pair of complex roots).

$\therefore \quad C.F = e^{-\frac{1}{2}x}\,(C_1 \cos\frac{\sqrt{3}}{2}x + C_2\ \sin\frac{\sqrt{3}}{2}x).$

$Now,\ \ P.I. = \frac{1}{(D^2+D+1)}(1-e^x)^2 = \frac{1}{(D^2+D+1)}\ (1-2e^x + e^{2x}).$

$\therefore \quad P.I. = \frac{1}{(D^2+D+1)}\ e^{0x} - 2\ \frac{1}{(D^2+D+1)}\ e^x + \frac{1}{(D^2+D+1)}\ e^{2x}.$

$\therefore \quad P.I. = \frac{1}{(0^2+0+1)}\ e^{0x} - 2\ \frac{1}{(1^2+1+1)}\ e^x + \frac{1}{(2^2+2+1)}\ e^{2x}.$

$\therefore \quad P.I. = 1 - \frac{2}{3}\ e^x + \frac{1}{7}\ e^{2x}.$

$\therefore \quad The\ \ solution\ is\ \ y = C.F + P.I.$

$\therefore \quad y = e^{-\frac{1}{2}x}\,(C_1 \cos\frac{\sqrt{3}}{2}x + C_2\ \sin\frac{\sqrt{3}}{2}x) + 1 - \frac{2}{3}\ e^x + \frac{1}{7}\ e^{2x}.$

12. Solve $(D^2+1)^2 y = x^4 + 2\sin x \cos 3x.$

Solution:

Given $(D^2+1)^2 y = x^4 + 2\sin x\cos 3x$..........(1).

The Auxiliary equation is $(m^2+1)^2 = 0.$ $\therefore$ $m^2+1 = 0$ and $m^2+1=0.$

$\therefore$ $m^2 = -1$ and $m^2 = -1.$ $\therefore$ $\mathrm{m} = \pm\,\mathrm{i}$ and $\mathrm{m} = \pm\,\mathrm{i}.$

$\therefore$ The roots are $\mathrm{m}_1,\ \mathrm{m}_2 = 0 \pm \mathrm{i} = \mathrm{a} \pm \mathrm{ib}$ and $\mathrm{m}_3,\ \mathrm{m}_4 = 0 \pm \mathrm{i} = \mathrm{a} \pm \mathrm{ib}.$
(A pair of complex roots repeated).

$\therefore \quad C.F = e^{0\,x}\,[(C_1 + C_2\,x)\cos x + (C_3 + C_4 x)\sin x].$

$\therefore \quad C.F = (C_1 + C_2\,x)\cos x + (C_3 + C_4 x)\sin x.$

$Now,\ \ P.I. = \frac{1}{(D^2+1)^2}[x^4 + 2\sin x\cos 3x]$. Using 2sinAcosB =sin(A+B) + sin(A−B),

We get, $P.I. = \frac{1}{(D^2+1)^2}[x^4 + sin\ 4x - \ sin\ 2x]$.

$\therefore \quad P.I. = \frac{1}{(D^2+1)^2}x^4 + \frac{1}{(D^2+1)^2}sin\ 4x - \frac{1}{(D^2+1)^2}\ sin\ 2x = \mathrm{P}_1 + \mathrm{P}_2 - \mathrm{P}_3.$

Now, $\mathrm{P}_1 = \frac{1}{(D^2+1)^2}x^4 = \frac{1}{D^4+2D^2+1}x^4 = \frac{1}{1+2D^2+D^4}x^4.$

$\therefore \quad \mathrm{P}_1 = x^4 - 24x^2 + 72.$

	$x^4 - 24x^2 + 72$
$1 + 2D^2 + D^4$	x^4
	$x^4 + 24x^2 + 24$
Sub.	$0 - 24x^2 - 24$
	$-24x^2 - 96$
Sub.	$0 + 72$
	72
Sub.	0

$P_2 = \frac{1}{(D^2+1)^2} sin\ 4x$. [Using rule 2, put $D^2 = -a^2 = -4^2 = -16$].

$\therefore$ $P_2 = \frac{1}{(-16+1)^2} sin\ 4x = \frac{1}{225} sin\ 4x$.

$P_3 = \frac{1}{(D^2+1)^2}\ sin\ 2x$. [Using rule 2, put $D^2 = -a^2 = -2^2 = -4$].

$\therefore$ $P_3 = \frac{1}{(-4+1)^2} sin\ 2x = \frac{1}{9} sin\ 2x$.

$\therefore\quad P.I. = \ x^4\ -\ 24x^2 + 72 + \frac{1}{225} sin\ 4x - \frac{1}{9} sin\ 2x.$

$\therefore\quad The\ \ solution\ is\ \ \ y = C.F + P.I.$

$\therefore$ y $=(C_1 + C_2\,x) \cos x + (C_3 + C_4 x) \sin x + x^4\ -\ 24x^2 + 72 + \frac{1}{225} sin\ 4x - \frac{1}{9} sin\ 2x.$

13. Solve y'' – 2y' +2y = x + e^x.

Solution:

Given $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = x + e^x$.

i.e., $(D^2 - 2D + 2)y = x\ +\ e^x$..........(1).

The Auxiliary equation is $m^2 - 2m + 2 = 0$.

$\therefore\ m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2\pm\sqrt{-4}}{2} = \frac{2 \pm i2}{2} = 1 \pm\ i.$

$\therefore$ The roots are $m_1,\ m_2 = 1\ \pm\ i = a \pm ib$.

$\therefore\quad C.F = e^{\,x}\,(C_1 \cos x + C_2 \sin x)$.

$Now,\ \ P.I. = \frac{1}{(D^2-2D+2)}[x\ +\ e^x] = \frac{1}{(D^2-2D+2)} x + \frac{1}{(D^2-2D+2)}\ e^x$

$\therefore\quad P.I. = \ P_1 + P_2$.

Now, $P_1 = \frac{1}{(D^2-2D+2)} x = \frac{x}{(2-2D+D^2)}$.

$$\begin{array}{r|l} & \frac{x}{2} + \frac{1}{2} \\ \hline 2 - 2D + D^2 & x \\ & x - 1 \\ \hline & \quad 1 \\ & \quad 1 \\ \hline & \quad 0 \end{array}$$

$\therefore\quad P_1 = \frac{x}{2} + \frac{1}{2}.$

$P_2 = \frac{1}{(D^2-2D+2)}\ e^x$ put $D = a = 1$

$\therefore\ P_2 = \frac{e^x}{1} = e^x$

$\therefore\ The\ \ solution\ is\ \ \ y = C.F. + P.I.$

$\therefore\ \ y = e^{x}\,(C_1 \cos x + C_2 \sin x) + \frac{x}{2} + \frac{1}{2} + e^x$

14. Solve $(D^3 - 5D^2 + 7D - 3)y = e^{2x} \cosh x$.

Solution:

Given $(D^3 - 5D^2 + 7D - 3)\,y = e^{2x} \cosh x$……… (1). Where $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$,…..

$\therefore$ The auxiliary equation of (1) is $m^3 - 5m^2 + 7m - 3 = 0$.

By inspection method m = 1 is the root. By synthetic division method, we get,

1	1	−5	7	−3
	0	1	−4	3
	1	−4	3	0

$\therefore\ m^2 - 4m + 3 = 0.\ \therefore\ (m-1)(m-3) = 0.\quad \therefore\ m = 1,\ m = 3.$

$\therefore$ The roots are $m_1 = 1$, $m_2 = 1$, repeated and $m_3 = 3$.

$\therefore\ C.F = (C_1 + C_2 x)e^x + C_3 e^{3x}.$

$Now,\ \ P.I. = \frac{1}{(D^3 - 5D^2 + 7D - 3)}\,[\,e^{2x}\cosh x] = \frac{1}{(D^3 - 5D^2 + 7D - 3)}\,[\,e^{2x}\left(\frac{e^x + e^{-x}}{2}\right)]$

$\therefore\ P.I. = \frac{1}{2}\frac{1}{(D^3 - 5D^2 + 7D - 3)}\ e^{3x} + \frac{1}{2}\frac{1}{(D^3 - 5D^2 + 7D - 3)}\ e^x.$

$\therefore\ P.I. = \frac{x}{2}\frac{1}{(3D^2 - 10D + 7)}\ e^{3x} + \frac{x}{2}\frac{1}{(3D^2 - 10D + 7)}\ e^x.$

$\therefore\ P.I. = \frac{x}{2}\frac{1}{(27 - 30 + 7)}\ e^{3x} + \frac{x^2}{2}\frac{1}{(6D - 10)}\ e^x = \frac{x}{2}\frac{1}{(4)}\ e^{3x} + \frac{x^2}{2}\frac{1}{(6 - 10)}\ e^x.$

$\therefore\ P.I. = \frac{x}{8}e^{3x} - \frac{x^2}{8}e^x$

$\therefore\ The\ \ solution\ is\ \ \ y = C.F. + P.I.$

$\therefore\ \ y = (C_1 + C_2 x)e^x + C_3 e^{3x} + \frac{x}{8}e^{3x} - \frac{x^2}{8}e^x.$

HOME WORK:

Solve the following:

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{2x} + sinx + x.$ 2. $(D^2 - 1)y = cosx.$

3. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x + sin2x.$ 4. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x.$

5. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4cos^2x.$ 6. $(D - 2)^2y = 8(e^{2x} + sin2x + x^2).$

7. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x}.$ 8. $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x.$

9. $\frac{d^2y}{dx^2} - 4y = x$ 10. $(D^4 - 1)y = e^t.$

Method of Variation of parameters:

This method is generally applied to the equation of the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q\,y = R$, where P, Q, R, are function of x or constants.

Working Rule:

Suppose the given equation is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q\,y = R$....(1), where P and Q are constants.

1. First find the C.F of the equation (1) and write down it in the form C.F. $= C_1 U + C_2 V$, where C_1, and C_2 are constants, U and V are functions of x.

2. Assume that $y = A U + B V$……. (2) as the complete solution of the equation (1), where A, B are functions of x to be determined.

3. Find $W = \begin{vmatrix} U & V \\ U' & V' \end{vmatrix} = UV' - VU'$, where W is called Wronskian of U and V.

4. Find A & B by using the formulae, $A = -\int \frac{VR}{W} dx + C_1$ and $B = \int \frac{UR}{W} dx + C_2$.

5. Substitute A and B in equation(2).

Problems:

1. Using the method of variation of parameter solve $\frac{d^2y}{dx^2} + 4y = tan2x.$

Solution:

Given $(D^2 + 4)y = \tan 2x$..............(1)

AE is $m^2 + 4 = 0 \implies m = \pm 2i.$ $\therefore$ $C.F. = C_1 \cos 2x + C_2 \sin 2x.$

Take $U = \cos 2x,$ $V = \sin 2x,$ and $R = \tan 2x.$

$\therefore$ $U' = -2\sin 2x,$ $V' = 2\cos 2x.$

$\therefore \quad W = \begin{vmatrix} U & V \\ U' & V' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\,(cos^2 2x + sin^2 2x) = 2.$

$\therefore \quad A = -\int \frac{VR}{W}\,dx + c_1 = -\int \frac{\sin 2x \tan 2x}{2}\,dx + c_1 = -\int \frac{sin^2 2x}{2\cos 2x}\,dx + c_1.$

$\therefore \quad A = -\int \frac{1-cos^2 2x}{2\cos 2x}\,dx + c_1 = \int \frac{cos^2 2x - 1}{2\cos 2x}\,dx + c_1 = \int \frac{\cos 2x}{2} - \frac{\sec 2x}{2}\,dx + c_1.$

$\therefore \quad A = \left[\frac{\sin 2x}{4} - \frac{\log(\sec 2x + \tan 2x)}{4}\right] + c_1$

$B = \int \frac{UR}{W}\,dx + c_2 = \int \frac{\cos 2x \tan 2x}{2}\,dx + c_2 = \int \frac{\sin 2x}{2}\,dx + c_2 = \frac{-\cos 2x}{4} + c_2.$

Therefore the complete solution is $y = AU + BV$.

$y = \left[\frac{\sin 2x}{4} - \frac{\log(\sec 2x + \tan 2x)}{4} + c_1\right] \cos 2x + \left[\frac{-\cos 2x}{4} + c_2\right] \sin 2x.$

$\therefore \quad y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \, \log(\sec 2x + \tan 2x).$

2. using the method of variation of parameter solve $\frac{d^2y}{dx^2} + a^2 y = cosecax$

Solution :

Given $(D^2 + a^2)y = cosecax$..........(1)

AE is $m^2 + a^2 = 0 \implies m = \pm ai.$ $\quad \therefore \quad C.F. = C_1 \cos ax + C_2 \sin ax$

Take $U = \cos ax$, $\quad V = \sin ax$, $\quad$ and $\quad R = osecax$.

$\therefore \quad U' = -a \sin ax, \qquad V' = a \cos ax.$

$\therefore \quad W = \begin{vmatrix} U & V \\ U' & V' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a\,(cos^2 ax + sin^2 ax) = a.$

$\therefore \quad A = -\int \frac{VR}{W}\,dx + c_1 = -\int \frac{\sin ax \,\text{cosec}\, ax}{a}\,dx + c_1 = -\int \frac{1}{a}\,dx + c_1.$

$\therefore \quad A = -\frac{x}{a} + c_1.$

$B = \int \frac{UR}{W}\,dx + c_2 = \int \frac{\cos ax \; cosecax}{2}\,dx + c_2 = \int \frac{\cot ax}{a}\,dx + c_2 = \frac{\log \sin ax}{a^2} + c_2.$

Therefore the complete solution is $y = AU + BV$.

$\therefore \quad y = \left[-\frac{x}{a} + c_1\right] \cos ax + \left[\frac{\log \sin ax}{a^2} + c_2\right] \sin ax.$

$\therefore \quad y = c_1 \cos ax + c_2 \sin ax - \frac{x \cos ax}{a} + \frac{(\sin ax) \log \sin ax}{a^2}$

3. Using the method of variation of parameter solve $\frac{d^2y}{dx^2}$ - 6 $\frac{dx}{dy} + 9y = \frac{e^{3x}}{x^2}$.

Solution:

Given $(D^2-6D+9)y=\frac{e^{3x}}{x^2}$ .... (1).

AE is $m^2-6m+9=0 \implies m=3,3.$ $\therefore$ $C.F.=(C_1+C_2x)e^{3x} = C_1e^{3x}+C_2xe^{3x}.$

Take $U=e^{3x}$, $V=xe^{3x}$, and $R=\frac{e^{3x}}{x^2}$.

$\therefore$ $U'=3e^{3x}$, $V'=3x\,e^{3x}+e^{3x}.$

$\therefore$ $W=\begin{vmatrix} U & V \\ U' & V' \end{vmatrix}=\begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3x\,e^{3x}+e^{3x} \end{vmatrix}=3x\,e^{6x}+e^{6x}-3x\,e^{3x}=e^{6x}.$

$\therefore$ $A=-\int\frac{VR}{W}dx+c_1=-\int\frac{x\,e^{3x}}{e^{6x}}.\frac{e^{3x}}{x^2}dx+c_1=-\int\frac{1}{x}dx+c_1.$

$\therefore$ $A=-\log x+c_1.$

$B=\int\frac{UR}{W}dx+c_2=\int\frac{e^{3x}}{e^{6x}}.\frac{e^{3x}}{x^2}dx+c_2=\int\frac{1}{x^2}dx+c_2=-\frac{1}{x}+c_2.$

Therefore the complete solution is $y=AU+BV$.

$\therefore$ $y=(-\log x+c_1)e^{3x}+(-\frac{1}{x}+c_2)x\,e^{3x}.$

$\therefore$ $y=(C_1+C_2x)e^{3x}-e^{3x}\log x-e^{3x}.$

4. Using the method of variation of parameter solve $y''-2y'+y=e^x\log x$.

Solution:

Given $(D^2-2D+1)y=e^x\log x$ .... (1).

AE is $m^2-2m+1=0 \implies m=1,1.$ $\therefore$ $C.F.=(C_1+C_2x)e^x.$

Take $U=e^x$, $V=x\,e^x$, and $R=e^x\log x$

$\therefore$ $U'=e^x$ $V'=xe^x+e^x.$

$\therefore$ $W=\begin{vmatrix} U & V \\ U' & V' \end{vmatrix}=\begin{vmatrix} e^x & xe^x \\ e^x & xe^x+e^x \end{vmatrix}=x\,e^x+e^x-x\,e^x=e^x.$

$\therefore$ $A=-\int\frac{VR}{W}dx+c_1=-\int\frac{x\,e^x\,e^x\log x}{e^{2x}}dx+c_1=-\int x\log x\,dx+c_1.$

$\therefore$ $A=-\left[\log x.\frac{x^2}{2}-\int\frac{x^2}{2}.\frac{1}{x}dx\right]+c_1=-\frac{x^2\log x}{2}+\frac{x^2}{4}+C_1.$

$B=\int\frac{UR}{W}dx+c_2=\int\frac{e^xe^x\log x}{e^{2x}}dx+c_2=\int\log x.\,1\,dx+c_2=\log x.x-\int x.\frac{1}{x}dx+c_2.$

$\therefore$ $B=x\log x-x+C_2$

Therefore the complete solution is $y=AU+BV$.

$\therefore \quad y = \left(-\frac{x^2 \log x}{2} + \frac{x^2}{4} + C_1\right) e^x + (x \log x - x + C_2)\, x\, e^x.$

$\therefore \quad y = (C_1 + C_2 x)\, e^x + \frac{x^2\, e^x \log x}{2} - \frac{3x^2 e^x}{4}.$

5. Using the method of variation of parameter solve $y'' - 3y' + 2y = \frac{1}{1+e^{-x}}$

Solution:

Given $(D^2 - 3D + 2)y = \frac{1}{1+e^{-x}}$ ....(1)

AE is $m^2 - 3m + 2 = 0 \Rightarrow m = 2, 1 \quad \therefore\ C.F. = C_1 e^x + C_2 e^{2x}.$

Take $U = e^x, \quad V = e^{2x},$ and $R = \frac{1}{1+e^{-x}}$.

$\therefore \quad U' = e^x \qquad V' = 2e^{2x}.$

$\therefore \quad W = \begin{vmatrix} U & V \\ U' & V' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}.$

$\therefore \quad A = -\int \frac{VR}{W} dx + c_1 = -\int \frac{e^{2x}}{e^{3x}} \frac{1}{1+e^{-x}}\, dx + c_1 = \int \frac{-e^{-x}}{1+e^{-x}} dx + c_1.$

$\therefore \quad A = \log(1+e^{-x}) + c_1 = \log\left(\frac{e^x+1}{e^x}\right) + c_1 .$

$B = \int \frac{UR}{W} dx + c_2 = \int \frac{e^x}{e^{3x}} \frac{1}{1+e^{-x}}\, dx + c_2 = \int \frac{1}{e^x(e^x+1)}\, dx + c_2.$

Take $e^x = t. \quad \therefore\ e^x\, dx = dt. \quad \therefore\ dx = \frac{dt}{e^x} = \frac{1}{t} dt.$

$\therefore \quad B = \int \frac{dt}{t^2(t+1)} + c_2.$ Consider $\frac{1}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1}.$

$\therefore \quad 1 = At(t+1) + B(t+1) + Ct^2 \Rightarrow A = -1,\ B = 1,\ C = 1.$

$\therefore \quad \frac{1}{t^2(t+1)} = \frac{-1}{t} + \frac{1}{t^2} + \frac{1}{t+1}.$

$\therefore \quad B = \int \left[\frac{-1}{t} + \frac{1}{t^2} + \frac{1}{t+1}\right] dt + c_2 = -\log t - \frac{1}{t} + \log(t+1) + c_2.$

$\therefore \quad B = \log\left(\frac{t+1}{t}\right) - \frac{1}{t} + c_2 = \log\left(\frac{e^x+1}{e^x}\right) - \frac{1}{e^x} + c_2.$

Therefore the complete solution is $y = AU + BV.$

$\therefore \quad y = \left(\log\left(\frac{e^x+1}{e^x}\right) + c_1\right) e^x + \left(\log\left(\frac{e^x+1}{e^x}\right) - \frac{1}{e^x} + c_2\right) e^{2x}.$

$\therefore \quad y = c_1 e^x + c_2 e^{2x} - e^x + e^x \log\left(\frac{e^x+1}{e^x}\right) + e^{2x} \log\left(\frac{e^x+1}{e^x}\right).$

6. Using the method of variation of parameter solve $\frac{d^2y}{dx^2} + y = x\sin x.$

Solution:

Given $(D^2+1)y = xsinx$ ……….(1)

AE is $m^2+1=0 \Longrightarrow m=\pm i$ $\therefore$ $C.F.= c_1\cos x + c_2 sinx.$

Take $U=\cos x,$ $V=sinx,$ and $\mathrm{R}=x\ sinx$.

$\therefore$ $U' = -sinx,$ $V' = \cos x.$

$\therefore$ $\mathrm{W}=\begin{vmatrix} U & V \\ \mathrm{U'} & \mathrm{V'} \end{vmatrix} = \begin{vmatrix} \cos x & sinx \\ -sinx & \cos x \end{vmatrix} = cos^2x + sin^2x = 1.$

$\therefore$ $A=-\int \frac{VR}{\mathrm{W}}\,\mathrm{dx} + c_1 = -\int \frac{\sin x.\ x\sin x}{1} dx + c_1 = -\int x\, sin^2x\, dx + c_1.$

$\therefore$ $A=-\int x\,sin^2x\,dx + c_1 = -\int x\left(\frac{1-\cos 2x}{2}\right)dx + c_1 = -\frac{1}{2}\int(x - x\cos 2x)\,dx + c_1$.

$\therefore$ $A=-\frac{1}{2}\left[\frac{x^2}{2} - \left\{x\left(\frac{sin2x}{2}\right) - 1\left(\frac{-cos2x}{4}\right)\right\}\right] + c_1 = -\frac{x^2}{4} + \frac{x}{4}sin2x + \frac{1}{8}cos2x + c_1$.

$\mathrm{B}=\int \frac{UR}{\mathrm{W}}\,\mathrm{dx} + c_2 = \int \cos x.\ x\ sinx\ dx + c_2 = \int x\sin x\cos x\,dx + c_2.$

$\therefore$ $\mathrm{B}=\int x\left(\frac{sin2x}{2}\right)dx + c_2 = \frac{1}{2}\int x\sin 2x\,dx + c_2 = \frac{1}{2}\left[x\left(\frac{-cos2x}{2}\right) - 1\left(\frac{-sin2x}{4}\right)\right] + c_2.$

$\therefore$ $\mathrm{B}=-\frac{x}{4}cos2x + \frac{1}{8}sin2x + c_2.$

Therefore the complete solution is $y = AU + BV.$

$\therefore$ $\mathrm{y}=\left(-\frac{x^2}{4} + \frac{x}{4}sin2x + \frac{1}{8}cos2x + c_1\right)\cos x + \left(-\frac{x}{4}cos2x + \frac{1}{8}sin2x + c_2\right)sinx.$

$\therefore$ $\mathrm{y}=c_1 cos\ x + c_2 sin\ x - \frac{x^2}{4}\cos x + \left(\frac{x}{4}sin2x + \frac{1}{8}cos2x\right)cos\ x - \left(\frac{x}{4}cos2x - \frac{1}{8}sin2x\right)sin\ x.$

7. Using the method of variation of parameter solve $y'' - 2y' + 2y = e^x\ tanx.$

Solution:

Given $(D^2 - 2D + 2)y = e^x\ tanx$ ……….(1)

AE is $m^2 - 2m + 2 = 0 \Longrightarrow m = \frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm\sqrt{-4}}{2} = \frac{2\pm i2}{2} = 1 \pm i.$

$\therefore$ The roots are $\mathrm{m}_1, \mathrm{m}_2 = 1 \pm \mathrm{i} = \mathrm{a} \pm \mathrm{ib}.$

$\therefore$ $C.F = e^x(C_1\cos x + C_2\sin x) = C_1 e^x\cos x + C_2 e^x\sin x.$

Take $U = e^x\cos x,$ $V = e^x sinx,$ and $\mathrm{R} = e^x\ tanx.$

$\therefore$ $U' = -e^x sinx + e^x\cos x,$ $V' = e^x\cos x + e^x sinx.$

$\therefore$ $\mathrm{W}=\begin{vmatrix} U & V \\ \mathrm{U'} & \mathrm{V'} \end{vmatrix} = \begin{vmatrix} e^x\cos x & e^x sinx \\ -e^x sinx + e^x\cos x & e^x\cos x + e^x sinx \end{vmatrix}$

$\therefore \quad W = e^{2x}(\cos^2 x + \sin x \cos x + \sin^2 x - \sin x \cos x) = e^{2x}$.

$\therefore \quad A = -\int \frac{VR}{W}\, dx + c_1 = -\int \frac{e^x \sin x . e^x \tan x}{e^{2x}} dx + c_1 = -\int \frac{\sin^2 x}{\cos x} dx + c_1.$

$\therefore \quad A = -\int \frac{(1-\cos^2 x)}{\cos x} dx + c_1 = -\int (\sec x - \cos x)\, dx + c_1$

$\therefore \quad A = -[\log(\sec x + \tan x) - \sin x] + c_1$.

$B = \int \frac{UR}{W}\, dx + c_2 = \int \frac{e^x \cos x . e^x \tan x}{e^{2x}} dx + c_2 = \int \sin x\, dx + c_2 = -\cos x + c_2.$

Therefore the complete solution is $y = AU + BV$.

$\therefore \quad y = [-(\log(\sec x + \tan x) + \sin x) + c_1]\, e^x \cos x + (-\cos x + c_2) e^x \sin x.$

$\therefore \quad y = e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x . \log(\sec x + \tan x).$

8. Using the method of variation of parameter solve $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$.

Solution:

Given $(D^2 + 1)y = \frac{1}{1+\sin x}$ ....(1)

AE is $m^2 + 1 = 0 \Longrightarrow m = \pm i$.

$\therefore$ The roots are $m_1, m_2 = 0 \pm i = a \pm ib$. $\quad \therefore \; C.F = C_1 \cos x + C_2 \sin x).$

Take $U = \cos x, \quad V = \sin x,$ and $R = \frac{1}{1+\sin x}$.

$\therefore \quad U' = -\sin x, \quad V' = \cos x.$

$\therefore \quad W = \begin{vmatrix} U & V \\ U' & V' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$

$\therefore \quad A = -\int \frac{VR}{W}\, dx + c_1 = -\int \frac{\sin x.}{1} . \frac{1}{1+\sin x} dx + c_1 = -\int \frac{\sin x(1-\sin x)}{(1+\sin x)(1-\sin x)} dx + c_1.$

$\therefore \quad A = -\int \frac{\sin x - \sin^2 x)}{(1-\sin^2 x)} dx + c_1 = -\int \frac{\sin x - \sin^2 x)}{\cos^2 x} dx + c_1 = -\int (\sec x \tan x - \tan^2 x)\, dx + c_1$.

$\therefore \quad A = -\int [\sec x \tan x - (\sec^2 x - 1)] dx + c_1 = -\sec x + \tan x - x + c_1$.

$B = \int \frac{UR}{W}\, dx + c_2 = \int \frac{\cos x}{1+\sin x} dx + c_2 = \log(1 + \sin x) + c_2$.

Therefore the complete solution is $y = AU + BV$.

$\therefore \quad y = (-\sec x + \tan x - x + c_1)\cos x + (\log(1 + \sin x) + c_2)\sin x.$

$\therefore \quad y = c_1 \cos x + c_2 \sin x - 1 + \sin x - x\cos x - \sin x . \log(1 + \sin x).$

HOME WORK:

Solve the following by using the method of variations of parameters :

1. $\frac{d^2y}{dx^2}+y=secx.$ 2. $\frac{d^2y}{dx^2}+y=tanx.$ 3. $\frac{d^2y}{dx^2}-2\frac{dx}{dy}=e^x \sin x.$

4. $y''+4y=4\sec^2 2x.$ 5. $\frac{d^2y}{dx^2}-y=\frac{1}{1+e^x}$. 6. $\frac{d^2y}{dx^2}+y=secx\ tanx.$

7. $y''+2y'+y=e^{-x}\log x.$ 8. $\frac{d^2y}{dx^2}+y=cosecx\ cotx.$

Linear Differential Equations with variable Coefficients:

Cauchy's homogeneous Linear Equations:

An equation of the form $x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}\ldots\ldots\ldots\ldots+a_ny=\varphi(x)$ is called Cauchy's homogeneous Linear Equations.

This equation can be reducible to linear differential equation with constant coefficients, by taking substitution $x=e^t$ i.e., $t=logx.$

Then $x\frac{dy}{dx}=Dy,\quad x^2\frac{d^2y}{dx^2}=D(D-1)y,$

$x^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y$ ... etc. Where $D=\frac{d}{dt}$

Problems:

1. Solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0.$

Solution:

Given $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$....... (1)

$Take\ \ t=\log x\ \ or\ x=e^t.\ \ \therefore\ x\frac{dy}{dx}=Dy,\qquad x^2\frac{d^2y}{dx^2}=D(D-1)y.$ Where $D=\frac{d}{dt}$

$Then\ given\ equation\ (1)\ reduces\ to$

$D(D-1)y+Dy-y=0.\ \ \therefore\ (D^2-1)y=0.$

$\therefore$ AE is $m^2-1=0.$ $\therefore$ m = 1, -1. $\therefore$ C.F. $=c_1e^t+c_2e^{-t}.$

$\therefore$ The general solution is $y=c_1e^t+c_2e^{-t}.\ Put\ t=logx\ \ or\ \ x=e^t.\ \ \therefore\ y=c_1x+\frac{c_2}{x}.$

2. $solve\ x^2y''-2xy'-4y=x^4.$

Solution:

Given $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^4.$

$Put\ t=\log x\ \ or\ x=e^t.\ \ \therefore\ x\frac{dy}{dx}=Dy,\qquad x^2\frac{d^2y}{dx^2}=D(D-1)y.$ Where $D=\frac{d}{dt}$.

$Then\ given\ equation\ (1)\ reduces\ to$

$D(D-1)y-2Dy-4y=(e^t)^4.\ \therefore\ (D^2-3D-4)y=e^{4t}.$

$\therefore$ AE is $m^2-3m-4=0.\ \therefore\ (m-4)(m+1)=0.$

$\therefore\ m=4,\ m=-1.\quad \therefore\ CF=c_1e^{4t}+c_2e^{-t}.$

$PI=\frac{e^{4t}}{D^2-3D-4}=\frac{te^{4t}}{2D-3}=\frac{te^{4t}}{5}.$

$\therefore$ The general solution is $y=C.F.+P.I.$

$\therefore\ y=c_1e^{4t}+c_2e^{-t}+\frac{te^{4t}}{5}.$

$Put\ t=logx\ or\ x=e^t.\quad \therefore\ y=c_1x^4+\frac{c_2}{x}+\frac{x^4\log x}{5}.$

3. Solve $x^2y''+xy'+y=2cos^2(logx)$

Solution:
This is a Cauchy's homogeneous linear equation

Put $x=e^t,\ i.e.,\ t=\log x,$ so that $x\frac{dy}{dx}=Dy\,,x^2\frac{d^2y}{dx^2}=D(D-1)y$ where $D=\frac{d}{dt}.$

Then the given equation becomes $[D(D-1)+D+1]y=2cos^2t.$

$or\ (D^2+1)y=2cos^2t$ ....(1)

Which is a linear equation with constant coefficients.

Its A.E is (D^2+1)=0 hence D=$\pm i.\quad \therefore C.F=(c_1\cos t+c_2\sin t).$

And P.I=$\frac{1}{(D^2+1)}2cos^2t=\frac{1}{(D^2+1)}(1+cos2t)=\frac{1}{(D^2+1)}e^{0x}+\frac{1}{(D^2+1)}(cos2t)=1-\frac{\cos 2t}{3}.$

Hence the solution of (1) is $=(c_1\cos t+c_2\sin t)+1-\frac{\cos 2t}{3}.$

$y=(c_1\cos\log x+c_2\sin\log x)+1-\frac{\cos 2(\log x)}{3}.$

4. Solve $x\frac{d^2y}{dx^2}-\frac{2y}{x}=x+\frac{1}{x^2}.$

Solution:
Multiplying given equation by x, it becomes, $x^2y''-2y=x^2+\frac{1}{x}$ ....(1)

Put $x=e^t,\ i.e.,\ t=\log x,$ so that $x\frac{dy}{dx}=Dy,\ x^2\frac{d^2y}{dx^2}=D(D-1)y.$ Where $D=\frac{d}{dt}.$

Then equation (1) becomes $(D^2-D-2)y=e^{2t}+e^{-t}$

For this the A. E is m^2-m-2=0. Hence m =2 ,−1.

$\therefore C.F=c_1e^{2t}+c_2e^{-t}$

And P.I$=\frac{1}{(D^2-D-2)}(e^{2t}+e^{-t})=\frac{te^{2t}}{3}-\frac{te^{-t}}{3}$.

Hence general solution is $y=c_1e^{2t}+c_2e^{-t}+\frac{te^{2t}}{3}-\frac{te^{-t}}{3}$.

Or $y=c_1x^2+c_2\frac{1}{x}+\frac{\log x}{3}x^2-\frac{\log x}{3x}$

5. *Solve* $x^2y''-xy'+y=\log x$.

Solution:

we have $x^2y''-xy'+y=logx$

Put $t=\log x$ *or* $x=e^t$. $\therefore$ $xy'=Dy,\ x^2y''=D(D-1)y$. Where D $=\frac{d}{dt}$.

Then given equation reduces to $D(D-1)y-Dy+y=t$.

$\therefore$ $(D^2-2D+1)y=t$. $\therefore$ AE is $m^2-2m+1=0$. $\therefore$ $(m-1)^2=0$. $\therefore$ $m=1,1$.

$\therefore$ $CF=(c_1+c_2t)e^t$.

$PI=\frac{t}{(D-1)^2}=(1-D)^{-2}t=(1+2D+3D^2\ldots)t$. $\therefore$ $PI=t+2$.

The general solution is given by $y=CF+PI$.

$\therefore$ $y=(c_1+c_2t)e^t+t+2$.

Put $t=logx$ *or* $x=e^t$. $\therefore$ $y=(c_1+c_2\log x)x+\log x+2$.

6. Solve $x^2y''-3xy'+4y=1+x^2$.

Solution:

Given $x^2y''-3xy'+4y=1+x^2$.

Put $t=\log x$ *or* $x=e^t$. $\therefore$ $xy'=Dy$ $,x^2y''=D(D-1)y$. Where $D=\frac{d}{dt}$.

Then given equation reduces to $D(D-1)y-3Dy+4y=1+(e^t)^2$.

$\therefore$ $(D^2-4D+4)y=1+e^{2t}$. $\therefore$AE is $m^2-4m+4=0$. $\therefore$ $(m-2)^2=0$.

$\therefore$ $m=2,2$. $\therefore$ $CF=(c_1+c_2t)e^{2t}$.

$PI=\frac{1+e^{2t}}{D^2-4D+4}=\frac{1}{(D-2)^2}+\frac{e^{2t}}{(D-2)^2}=\frac{1}{4}+\frac{t^2e^{2t}}{2}$.

$\therefore$ The general solution is given by $y=CF+PI$.

$\therefore$ $y=(c_1+c_2t)e^{2t}+\frac{1}{4}+\frac{t^2e^{2t}}{2}$. *Put* $t=logx$ *or* $x=e^t$.

$\therefore$ $y=(c_1+c_2\log x)x^2+\frac{1}{4}+\frac{x^2(\log x)^2}{2}$.

7. Solve $x^2y''-3xy'+5y=3\sin(logx)$.

Solution:
Given $x^2y'' - 3xy' + 5y = 3\sin(logx)$.

$Put\ t = \log x\ \ or\ x = e^t.\ \therefore\ \ xy' = Dy,\ \ x^2y'' = D(D-1)y.$ Where $D = \frac{d}{dt}$.

$Then\ given\ equation\ (1)\ reduces\ to\ D(D-1)y - 3Dy + 5y = 3sint.$

$\therefore (D^2 - 4D + 5)y = 3sint.\ \ \therefore$ AE is given by $m^2 - 4m + 5 = 0$.

$\therefore\ m = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{4\pm\sqrt{16-20}}{2} = \frac{4\pm 2i}{2} = 2 \pm i.\ \ \ \therefore\ CF = (c_1cost + c_2\sin t)e^{2t}.$

$PI = \frac{3sint}{D^2-4D+5} = \frac{3sint}{4(1-D)} = \frac{3(D+1)\ sint}{4(1-D^2)} = \frac{3cost+3sint}{4(1-D^2)} = \frac{3(cost+sint)}{8}.$

$\therefore y = CF + PI.\ \ \therefore\ \ y = (c_1cost + c_2\sin t)e^{2t} + \frac{3(cost+sint)}{8}.$

$Put\ t = logx\ or\ x = e^t.$

$\therefore\ \ y = [c_1\cos(\log x) + c_2\sin(logx)]\ \ x^2 + \frac{3[\cos(logx)+\sin(logx)]}{8}.$

8. Solve $x^2y'' + xy' + 9y = 3x^2 + \sin(3\log x)$

Solution:
Given $x^2y'' + xy' + 9y = 3x^2 + \sin(3\log x)$.

$Put\ t = \log x\ \ or\ x = e^t.\ \ \ \therefore\ \ xy' = Dy,\ \ x^2y'' = D(D-1)y.$ Where $D = \frac{d}{dt}$.

$Then\ given\ equation\ reduces\ to\ D(D-1)y + Dy + 9y = 3e^{2t} + \sin(3t).$

$\therefore\ \ (D^2 + 9)y = 3e^{2t} + \sin(3t).\ \ \therefore$ AE is $m^2 + 9 = 0.\ \ \ \therefore\ \ m = \pm 3i.$

$\therefore\ CF = c_1cos3t + c_2sin3t.$

$PI = \frac{3e^{2t}}{D^2+9} + \frac{\sin(3t)}{D^2+9} = \frac{3e^{2t}}{13} + \frac{t\sin(3t)}{2D} = \frac{3e^{2t}}{13} + \frac{t}{2}\int sin3t\,dt = \frac{3e^{2t}}{13} - \frac{tcos3t}{6}.$

$\therefore\ \ y = CF + PI.$

$\therefore\ \ y = c_1cos3t + c_2sin3t + \frac{3e^{2t}}{13} - \frac{tcos3t}{6}\ .\ \ Put\ t = logx\ or\ x = e^t.$

$\therefore\ \ y = c_1\cos(3\log x) + c_2\sin(3\log x) + \frac{3x^2}{13} - \frac{logx\ cos(3logx)}{6}.$

9. Solve $x^2y'' - 3xy' + 4y = (1+x)^2$

Solution:
Given $x^2y'' - 3xy' + 4y = (1+x)^2$.

$Put\ t = \log x\ \ or\ x = e^t.\ \ \therefore\ xy' = Dy,\ x^2y'' = D(D-1)y.$ Where D $= \frac{d}{dt}$.

$Then\ given\ equation\ (1)\ reduces\ to\ D(D-1)y - 3Dy + 4y = (1+e^t)^2.$

$\therefore\ \ (D^2 - 4D + 4)y = 1 + e^{2t} + 2e^t.\ \ \ \therefore$ AE is $m^2 - 4m + 4 = 0$.

$\therefore (m-2)^2 = 0. \quad \therefore\ m = 2,\ 2. \quad \therefore\ CF = (c_1 + c_2 t)e^{2t}.$

$PI = \frac{1+e^{2t}+2e^t}{(D-2)^2} = \frac{1}{(D-2)^2} + \frac{e^{2t}}{(D-2)^2} + \frac{2e^t}{(D-2)^2} = \frac{1}{4} + \frac{te^{2t}}{2(D-2)} + 2e^t = \frac{1}{4} + \frac{t^2e^{2t}}{2} + 2e^t.$

$y = CF + PI\ .$

$\therefore\ y = (c_1 + c_2 t)e^{2t} + \frac{1}{4} + \frac{t^2 e^{2t}}{2} + 2e^t. \quad Put\ t = logx\ or\ x = e^t.$

$\therefore\ \ y = (c_1 + c_2 \log x)x^2 + \frac{1}{4} + \frac{x^2(logx)^2}{2} + 2x\ .$

10. Solve $\mathbf{xy''' + y'' = \frac{1}{x}}$

Solution:

Given $xy''' + y'' = \frac{1}{x}$. Multiply by x^2.

$\therefore\ x^3y''' + x^2y'' = x.\ \ Put\ t = \log x\ \ or\ x = e^t$

$\therefore\ \ x^3y''' = D(D-1)(D-2)y,\ x^2y'' = D(D-1)y.$ Where $D = \frac{d}{dt}$.

$Then\ given\ equation\ (1)\ reduces\ to\ \ D(D-1)(D-2)y + D(D-1)y = e^t.$

$\therefore (D^3 - 3D^2 + 2D + D^2 - D)y = e^t. \quad \therefore\ \ (D^3 - 2D^2 + D)y = e^t.$

$\therefore$ AE is $m^3 - 2m^2 + m = 0. \quad \therefore\ m(m^2 - 2m + 1) = 0.\ \therefore\ m = 0, (m-2)^2 = 0.$

$\therefore\ \ m = 0, 2, 2. \quad \therefore\ CF = c_1 + (c_2 + tc_3)e^{2t}.$

$PI = \frac{e^t}{D^3-2D^2+D} = \frac{te^t}{3D^2-4D+1} = \frac{t^2e^t}{6D-4} = \frac{t^2e^t}{2}\ .$

$\therefore$ The general solution is given by $\ y = CF + PI.$

$\therefore\ y = c_1 + (c_2 + tc_3)e^{2t} + \frac{t^2e^t}{2}. \quad Put\ t = logx\ or\ x = e^t.$

$\therefore\ y = c_1 + (c_2 + c_3 \log x)x^2 + \frac{x}{2}(logx)^2\ .$

11. Solve $\mathbf{x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos logx.}$

Solution:

Given $\ x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos logx.$

$Put\ t = \log x\ \ or\ x = e^t. \ \ \therefore\ \ x^3y''' = D(D-1)(D-2)y,\ x^2y'' = D(D-1)y, xy' = Dy.$

Where $D = \frac{d}{dt}$. $\quad Then\ given\ equation\ reduces\ to$

$D(D-1)(D-2)y + 3D(D-1)y + Dy + 8y = 65\ cost.$

$\therefore\ \ (D^3 - 3D^2 + 2D + 3D^2 - 3D + D + 8)y = 65cost.$

$\therefore\ (D^3 + 8)y = 65cost.\ \ \therefore$ AE is given by $m^3 + 8 = 0.$

By inspection one of the root is m = -2 and will find other two roots by synthetic division method.

-2	1	0	0	8
	0	-2	4	-8
	1	-2	4	0

$\therefore\ m^2 - 2m + 4 = 0.\ \therefore\ m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2\pm\sqrt{4-16}}{2} = \frac{2\pm 2\sqrt{3}}{2}.\ \ \therefore\ m = 1 \pm \sqrt{3}\ , m = -2.$

$\therefore CF = c_1 e^{-2t} + (c_2 cos\sqrt{3}\, t + c_3 sin\sqrt{3}\, t)e^t.$

$PI = \frac{65\, cost}{D^3+8} = \frac{65cost}{-D+8} = \frac{65cost(D+8)}{(64-D^2)} = \frac{65\,(-sint+8cost)}{64-D^2} = \frac{65(-\,sint+8cost)}{65} = -sint + 8cost.$

$\therefore$ The general solution is given by $y = CF + PI.$

$\therefore\ y = c_1 e^{-2t} + \left(c_2 cos\sqrt{3}t + c_3 sin\sqrt{3}t\right)e^t - sint + 8cost.$

$Put\ t = logx\ or\ x = e^t.$

$\therefore\ y = \frac{c_1}{x^2} + [c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x)]\, x - \sin(\log x) + 8\cos(\log x).$

Home work:

1. Solve $\boldsymbol{x^2y'' + xy' + 9y = 3x^3 + \sin(2logx)}$
2. Solve $\boldsymbol{x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)}$
3. Solve $\boldsymbol{x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x + logx}$

Legendre's linear differential Equations:

An equation of the form

$(ax+b)^n\frac{d^ny}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2(ax+b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} \ldots\ldots\ldots\ldots + a_n y = \varphi(x)$

is called the Legendre's linear differential Equations.

This equation can be reducible to linear differential equation with constant coefficients, by taking substitution, $ax + b = e^t \ \Longrightarrow\ t = log(ax+b).$

Then $(ax+b)\frac{dy}{dx} = aDy,\ (ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y,$

$(ax+b)^3\frac{d^3y}{dx^3} = a^3D(D-1)(D-2)y$ ...etc where $D = \frac{d}{dt}$

Problems:

1. **Solve $\boldsymbol{(1+x)^2y'' + (1+x)y' + y = 2\sin[\log(1+x)]}$**

Solution:

Given $(\boldsymbol{1+x})^2y'' + (1+x)y' + y = 2\sin[\log(1+x)]$

$Put\ t = \log(1+x)\ or\ 1 + x = e^t.\ \therefore\ (1+x)y' = 1.Dy,\ (1+x)^2y'' = 1^2.D(D-1)y.$

Where $= \frac{d}{dt}$. $Then\ given\ equation\ (1)\ reduces\ to\ \ D(D-1)y + Dy + y = 2sint.$

$\therefore$ $(D^2+1))y = 2sint.$ $\therefore$ AE is $m^2 + 1 = 0.$ $\therefore$ $m = \pm i.$ $\therefore$ $CF = c_1 \cos t + c_2 sint.$

$PI = \frac{2sint}{D^2+1} = \frac{2tsint}{2D} = t\int sint\ dt = -tcost.$

The general solution is given by $y = CF + PI.$ $\therefore$ $y = c_1 \cos t + c_2 \sin t - t\ cost$.

$Put\ t = \log(1+x)\ or\ 1 + x = e^t.$

$\therefore$ $y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \log(1+x) \cos[\log(1+x)]$.

2. Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin[\log(1+x)^2]$.

Solution:

Given $(1+x^2)y'' + (1+x)y' + y = \sin[2log(1+x)].$

This is a Legendre's linear equation. $\therefore$ Put $1 + x = e^t$, $i.e.$, $t = \log(1+x)$, so that

$(1+x)\frac{dy}{dx} = Dy$ and $(1+x)^2\frac{d^2y}{dx^2} = D(D-1)y$, where $D = \frac{d}{dt}$.

Then the given equation becomes $(D^2+1)y = \sin 2t$ ....(1).

Its A.E is $m^2 + 1 = 0.$ $\therefore$ $D = \pm i.$ $\therefore$ $C.F = c_1 \cos t + c_2 \sin t.$

And P.I=$\frac{1}{D^2+1}$ $(\sin 2t) = \frac{-1}{3}\sin\{2\log(1+x)\}.$

Therefore, the general solution of the given equation is y = CF + PI.

$\therefore$ $y = c_1 \cos t + c_2 \sin t - \frac{1}{3}\sin 2t.$

$\therefore$ $y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) - \frac{1}{3}\sin\{2\log(1+x)\}.$

3. Solve $(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 3(2x+1)$.

Solution:

Given $(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 3(2x+1).$

$Put\ t = \log(2x+1)\ or\ 1 + 2x = e^t.$

$\therefore$ $(1+2x)y' = 2.Dy$ $,(1+2x)^2y'' = 2^2 .\ D(D-1)y.$ Where $D = \frac{d}{dt}$.

$Then\ given\ equation\ reduces\ to\ 4D(D-1)y - 4Dy - 12y = 3e^t.$

$\therefore$ $(D^2 - 2D - 3)y = \frac{3e^t}{4}$. $\therefore$ AE is given by$m^2 - 2m - 3 = 0.$

$\therefore\ \ m^2-3m+m-3=0.\quad \therefore\ \ m(m-3)+1(m-3)=0.\ \therefore\ \ (m+1)(m-3)=0.$

$\therefore\ \ m=-1,\ m=3.\quad \therefore\ \ CF=c_1e^{-t}+c_2e^{3t}.$

$PI=\frac{3e^t}{4(D^2-2D-3)}=-\frac{3e^t}{16}.$ $\therefore$ The general solution is $y=CF+PI.$

$\therefore\ \ y=c_1e^{-t}+c_2e^{3t}-\frac{3e^t}{16}.$ $Put\ \ t=\log(1+2x)\ \ or\ \ 1+\ 2x=e^t.$

$\therefore\ \ y=c_1(2x+1)^{-1}+c_2(2x+1)^3-\frac{3(2x+1)}{16}.$

4. Solve $(2x+3)^2\frac{d^2y}{dx^2}+5(2x+3)\frac{dy}{dx}+y=4x.$

Solution:

Given $(2x+3)^2\frac{d^2y}{dx^2}+5(2x+3)\frac{dy}{dx}+y=4x.$

$Put\ t=\log(2x+3)\ \ or\ \ 3+\ 2x=e^t.$

$\therefore\ (3+2x)y'=2.Dy,\ \ (3+2x)^2y''=2^2.D(D-1)y.$ Where D $=\frac{d}{dt}.$

$Then\ given\ equation\ reduces\ to\ \ 4D(D-1)y+10Dy+y=\frac{4}{2}(e^t-3).$

$\therefore\ \ (4D^2+6D+1)y=2(e^t-3).\quad \therefore\ \ \left(D^2+\frac{3}{2}D+\frac{1}{4}\right)y=\frac{1}{2}(e^t-3).$

$\therefore$ AE is $m^2+\frac{3}{2}m+\frac{1}{4}=0.$ $\therefore\ \ m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-\frac{3}{2}\pm\sqrt{\frac{9}{4}-1}}{2}=\frac{-\frac{3}{2}\pm\sqrt{\frac{5}{4}}}{2}=\frac{-3\pm\sqrt{5}}{4}.$

$\therefore\ \ CF=c_1e^{\left(\frac{-3+\sqrt{5}}{4}\right)t}+c_2e^{\left(\frac{-3-\sqrt{5}}{4}\right)t}.$

$PI=\frac{(e^t-3)}{2(D^2+\frac{3}{2}D+\frac{1}{4})}=\frac{1}{2}\left(\frac{e^t}{D^2+\frac{3}{2}D+\frac{1}{4}}-\frac{3}{D^2+\frac{3}{2}D+\frac{1}{4}}\right)=\frac{1}{2}\left(\frac{4e^t}{11}-12\right).$

$\therefore$ The general solution is $y=CF+PI.$ $\therefore\ \ y=c_1e^{\left(\frac{-3+\sqrt{5}}{4}\right)t}+c_2e^{\left(\frac{-3-\sqrt{5}}{4}\right)t}+\frac{1}{2}\left(\frac{4e^t}{11}-12\right).$

$Put\ t=\log(2x+3)\ or\ 2x+3=e^t.$

$\therefore\ \ y=c_1(2x+3)^{\left(\frac{-3+\sqrt{5}}{4}\right)}+c_2(2x+3)^{\left(\frac{-3-\sqrt{5}}{4}\right)}+\frac{2}{11}(2x+3)-6.$

5. Solve $(2x-5)^2\frac{d^2y}{dx^2}-(2x-5)\frac{dy}{dx}-12y=6x^2.$

Solution:

Given $(2x-5)^2\frac{d^2y}{dx^2}-(2x-5)\frac{dy}{dx}-12y=6x^2.$
$Put\ t=\log(2x-5)\ \ or\ \ 2x-5=e^t.$

$\therefore\ \ (2x-5)y'=2.Dy\quad ,(2x-5)^2y''=2^2.\ D(D-1)y.$ Where $D=\frac{d}{dt}.$

Then given equation reduces to $4D(D-1)y-2Dy-12y=\frac{6(e^t+5)^2}{4}$.

$\therefore\ \left(D^2-\frac{3}{2}D-3\right)y=\frac{3(e^{2t}+25+2e^t)}{8}$. $\therefore$ AE is given by $m^2-\frac{3}{2}m-3=0$.

$\therefore\ m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{\frac{3}{2}\pm\sqrt{\frac{9}{4}+12}}{2}=\frac{\frac{3}{2}\pm\sqrt{\frac{57}{4}}}{2}=\frac{3\pm\sqrt{57}}{4}$. $\therefore\ CF=c_1e^{\left(\frac{3+\sqrt{57}}{4}\right)t}+c_2e^{\left(\frac{3-\sqrt{57}}{4}\right)t}$.

$PI=\frac{3(e^{2t}+25+10e^t)}{8(D^2-\frac{3}{2}D-3)}=\frac{3}{8}\left(\frac{e^{2t}}{(D^2-\frac{3}{2}D-3)}+\frac{25}{(D^2-\frac{3}{2}D-3)}+\frac{10e^t}{(D^2-\frac{3}{2}D-3)}\right)=\frac{3}{8}\left(\frac{e^{2t}}{-2}+\frac{25}{-3}-\frac{20e^t}{7}\right)$.

The general solution is $y=CF+PI$.

$\therefore\ y=\ c_1e^{\left(\frac{3+\sqrt{57}}{4}\right)t}+c_2e^{\left(\frac{3-\sqrt{57}}{4}\right)t}+\frac{3}{8}\left(\frac{e^{2t}}{-2}+\frac{25}{-3}-\frac{20e^t}{7}\right)$.

Put $t=\log(2x-5)$ *or* $2x-5=e^t$.

$\therefore\ y=c_1(2x-5)^{\left(\frac{3+\sqrt{57}}{4}\right)}+c_2(2x-5)^{\left(\frac{3-\sqrt{57}}{4}\right)}-\frac{3}{8}\left(\frac{(2x-5)^2}{2}+\frac{25}{3}+\frac{20(2x-5)}{7}\right)$.

6. Solve $(2x-1)^2\frac{d^2y}{dx^2}+(2x-1)\frac{dy}{dx}-2y=8x^2-2x+3$.

Solution:

Given $(2x-1)^2\frac{d^2y}{dx^2}+(2x-1)\frac{dy}{dx}-2y=8x^2-2x+3$.

This is a Legendre's linear equation. $\therefore$ Put $2x-1=e^t$, $i.e, t=\log(2x-1)$, so that

$(2x-1)\frac{dy}{dx}=2Dy$ and $(2x-1)^2\frac{d^2y}{dx^2}=4D(D-1)y$, where $D=\frac{d}{dt}$.

Then the given equation becomes $(4D^2-2D-2)y=2e^{2t}+3e^t+4$.

$\therefore\ (2D^2-D-1)y=e^{2t}+\frac{3}{2}e^t+2$ ....(1) $\therefore$ Its A.E is $2m^2-m-1=0$.

$\therefore\ m=1,\ \frac{-1}{2}$. $\therefore\ C.F=c_1e^t+c_2e^{-t/2}$.

And P.I=$\frac{1}{2D^2-D-1}\ (e^{2t}+\frac{3}{2}e^t+2)=\frac{1}{5}e^{2t}+\frac{t}{2}e^t-2$.

Hence the solution of (1) is $y=c_1e^t+c_2e^{-t/2}+\frac{1}{5}e^{2t}+\frac{t}{2}e^t-2$.

$\therefore\ y=c_1(2x-1)+c_2(2x-1)^{-1/2}+\frac{1}{5}(2x-1)^2+\frac{1}{2}(2x-1)^1\log(2x-1)-2$.

7. Solve $(x-1)^3\frac{d^3y}{dx^3}+2(x-1)^2\frac{d^2y}{dx^2}-4(x-1)\frac{dy}{dx}+4y=4log(x-1)$.

Solution:

Given $(x-1)^3\frac{d^3y}{dx^3}+2(x-1)^2\frac{d^2y}{dx^2}-4(x-1)\frac{dy}{dx}+4y=4log(x-1)$.

This is a Legendre's linear equation. $\therefore$ Put $x-1=e^t$, $i.e.,$ $t=\log(x-1)$, so that

$(x-1)\frac{dy}{dx}=Dy$, $(x-1)^2\frac{d^2y}{dx^2}=D(D-1)y$, and $(x-1)^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y$.

Where $=\frac{d}{dt}$. Then the given equation becomes $(D^3-D^2-4D+4)y=4t$........(1).

$\therefore$ Its A.E is $m^3-m^2-4m+4=0$. $\therefore$ $m=1\,,\ \pm 2$. $\therefore$ $C.F=c_1e^t+c_2e^{2t}+c_3e^{-2t}$.

And P.I=$\frac{1}{D^3-D^2-4D+4}$ $(4t)=$ t +1.

Therefore, the general solution of the given equation is

$y=\ c_1e^t+c_2e^{2t}+c_3e^{-2t}+$ t +1.

$\therefore$ $y=c_1(x-1)+c_2(x-1)^2+c_3(x-1)^{-2}+\log(x-1)+1$.

8. Solve $(2x+3)^2\frac{d^2y}{dx^2}+6(2x+3)\frac{dy}{dx}+6y=log(2x+3)$.

Solution:

Given $(2x+3)^2\frac{d^2y}{dx^2}+6(2x+3)\frac{dy}{dx}+6y=\log(2x+3)$.

$Put\ t=\log(2x+3)\quad or\ 2x+3=e^t$.

$\therefore$ $(2x+3)y'=2.Dy$, $(2x+3)^2y''=2^2.D(D-1)y$. Where $D=\frac{d}{dt}$.

$Then\ given\ equation\ reduces\ to$ $4D(D-1)y+12Dy+6y=t$.

$\therefore$ $(4D^2+8D+6)y=t$. $\therefore$ $(D^2+2D+\frac{3}{2})y=\frac{t}{4}$.

$\therefore$ AE is $m^2+2m+\frac{3}{2}=0$. $\therefore$ $m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{4-6}}{2}=\frac{-2\pm i\sqrt{2}}{2}=-1\pm\frac{i}{\sqrt{2}}$.

$\therefore$ $CF=[c_1\cos\frac{1}{\sqrt{2}}t+c_2\sin\frac{1}{\sqrt{2}}t]e^{-t}$.

$$PI=\frac{t}{4\left(D^2+2D+\frac{3}{2}\right)}=\frac{t}{6\left[1+\left(\frac{2D^2}{3}+\frac{4D}{3}\right)\right]}=\frac{t\left[1+\left(\frac{2D^2}{3}+\frac{4D}{3}\right)\right]^{-1}}{6}=\frac{t}{6}\left[1-\frac{2D^2}{3}-\frac{4D}{3}\right]=\frac{1}{6}\left[t-\frac{4}{3}\right].$$

$\therefore$ The general solution is given by $y=CF+PI$.

$\therefore$ $y=\left[c_1\cos\frac{1}{\sqrt{2}}t+c_2\sin\frac{1}{\sqrt{2}}t\right]e^{-t}+\frac{1}{6}\left[t-\frac{4}{3}\right]$. $Put\ t=\log(2x+3)\ or\ 2x+3=e^t$.

$\therefore$ $y=\left[c_1\cos\left(\frac{\log(2x+3)}{\sqrt{2}}\right)+c_2\sin\left(\frac{\log(2x+3)}{\sqrt{2}}\right)\right](2x+3)^{-1}+\frac{1}{6}\left[\log(2x+3)-\frac{4}{3}\right]$.

9. Solve $(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y=8x^2+4x+1$

Solution:

Given $(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y=8x^2+4x+1.$

$Put\ t=\log(3x+2)\quad or\ 3x+2=e^t.$

$\therefore\quad (3x+2)y'=3.Dy,\ (3x+2)^2y''=3^2.D(D-1)y.$ Where $D=\frac{d}{dt}$.

$Then\ given\ equation\ reduces\ to\ \ 9D(D-1)y+9Dy-36y=\frac{8(e^t-2)^2}{9}+\frac{4(e^t-2)}{3}+1.$

$\therefore\quad 9(D^2-4)y=\frac{(8e^{2t}-32e^t+32+12e^t-24+9}{9}.\quad \therefore\quad (D^2-4)y=\frac{(8e^{2t}-20e^t+17)}{81}.$

$\therefore$ AE is $m^2-4=0.\ \therefore\ m=\pm 2.\quad \therefore\ CF=c_1e^{2t}+c_2e^{-2t}.$

$PI=\frac{(8e^{2t}-20e^t+17)}{81(D^2-4)}=\frac{8e^{2t}}{81(D^2-4)}-\frac{20e^t}{81(D^2-4)}+\frac{17}{81(D^2-4)}=\frac{1}{81}\left[\frac{4te^{2t}}{D}+\frac{20e^t}{3}-\frac{17}{4}\right].$

$\therefore\quad PI=\frac{1}{81}\left[\frac{4te^{2t}}{2}+\frac{20e^t}{3}-\frac{17}{4}\right]=\frac{1}{81}\left[2te^{2t}+\frac{20e^t}{3}-\frac{17}{4}\right].$

$\therefore$ The general solution is given by $y=CF+PI.$

$\therefore\quad y=c_1e^{2t}+c_2e^{-2t}+\frac{1}{81}\left[2te^{2t}+\frac{20e^t}{3}-\frac{17}{4}\right].\ \ Put\ t=\log(3x+2)\ or\ 3x+2=e^t.$

$\therefore\quad y=c_1(3x+2)^2+c_2(3x+2)^{-2}+\frac{1}{81}\left[2\log(3x+2)\,(3x+2)^2+\frac{20(3x+2)}{3}-\frac{17}{4}\right].$

11. Solve $(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y=x^2+x+1.$

Solution:

Given $(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y=x^2+x+1.$

$Put\ t=\log(3x+2)\quad or\ 3x+2=e^t.$

$\therefore\quad (3x+2)y'=3.Dy\quad ,(3x+2)^2y''=3^2\,.\ D(D-1)y.$ Where $D=\frac{d}{dt}$.

$Then\ given\ equation\ reduces\ to\ \ 9D(D-1)y+15Dy-3y=\frac{(e^t-2)^2}{9}+\frac{(e^t-2)}{3}+1.$

$\therefore\quad 9(D^2+\frac{2}{3}D-\frac{1}{3})y=\frac{(e^{2t}-4e^t+4+3e^t-6+9}{9}.\quad \therefore\quad (D^2+\frac{2}{3}D-\frac{1}{3})y=\frac{(e^{2t}-e^t+7)}{81}.$

$\therefore$ AE is $m^2+\frac{2}{3}m-\frac{1}{3}=0.\quad \therefore\ m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-\frac{2}{3}\pm\sqrt{\frac{4}{9}+\frac{4}{3}}}{2}=\frac{-\frac{2}{3}\pm\sqrt{\frac{16}{9}}}{2}=\frac{-\frac{2}{3}\pm\frac{4}{3}}{2}.$

$\therefore\quad m=\frac{1}{3},\ -1.\quad \therefore\ CF=c_1e^{-t}+c_2e^{\frac{1}{3}t}.$

$PI=\frac{(e^{2t}-e^t+7)}{81\left(D^2+\frac{2}{3}D-\frac{1}{3}\right)}=\frac{e^{2t}}{81\left(D^2+\frac{2}{3}D-\frac{1}{3}\right)}-\frac{e^t}{81\left(D^2+\frac{2}{3}D-\frac{1}{3}\right)}+\frac{7}{81\left(D^2+\frac{2}{3}D-\frac{1}{3}\right)}=\frac{1}{81}\left[\frac{e^{2t}}{5}+\frac{3e^t}{4}-21\right].$

$\therefore\quad PI=\frac{1}{81}\left[\frac{e^{2t}}{5}+\frac{3e^t}{4}-21\right].\quad \therefore$ The general solution is given by $y=CF+PI.$

$\therefore \quad y = c_1 e^{-t} + c_2 e^{\frac{t}{3}} + \frac{1}{81}\left[\frac{e^{2t}}{5} + \frac{3e^t}{4} - 21\right]$. $\quad Put\ t = \log(3x+2)\ or\ 3x + 2 = e^t$.

$\therefore \quad y = c_1(3x+2)^{-1} + c_2(3x+2)^{\frac{1}{3}} + \frac{1}{81}\left[\frac{(3x+2)^2}{5} + \frac{3(3x+2)}{4} - 21\right]$.

12. Solve $(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x + 5$.

Solution:

Given $(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x + 5$.

$Put\ t = \log(2x+1)\ \ or\ \ 1 + 2x = e^t$.

$\therefore \ (1+2x)y' = 2.Dy,\ \ (1+2x)^2 y'' = 2^2.D(D-1)y$. Where $D = \frac{d}{dt}$.

$Then\ given\ equation\ reduces\ to\ \ 4D(D-1)y - 4Dy - 12y = 3(e^t - 1) + 5$.

$\therefore \quad (D^2 - 2D - 3)y = \frac{3e^t}{4} + \frac{1}{2}$. $\therefore$ AE is $m^2 - 2m - 3 = 0$. $\quad \therefore \quad m^2 - 3m + m - 3 = 0$.

$\therefore \quad m(m-3) + 1(m-3) = 0. \quad \therefore \quad (m+1)(m-3) = 0. \quad \therefore \quad m = -1, m = 3$.

$\therefore \quad CF = c_1 e^{-t} + c_2 e^{3t}$

$PI = \frac{3e^t}{4(D^2-2D-3)} + \frac{1}{2(D^2-2D-3)} = -\frac{3e^t}{16} - \frac{1}{6}$. $\quad \therefore$ The general solution is $y = CF + PI$.

$\therefore \ y = c_1 e^{-t} + c_2 e^{3t} - \frac{3e^t}{16} - \frac{1}{6}$. $\quad Put\ t = \log(1+2x)\ or\ 1 + 2x = e^t$.

$\therefore \quad y = c_1(2x+1)^{-1} + c_2(2x+1)^3 - \frac{3(2x+1)}{16} - \frac{1}{6}$.

13. Solve $(2x+1)^2 \frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$.

Solution:

Given $(2x+1)^2 \frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$.

$Put\ t = \log(2x+1)\ \ or\ \ 1 + 2x = e^t$.

$\therefore \ (1+2x)y' = 2.Dy,\ (1+2x)^2 y'' = 2^2 . D(D-1)y$. Where $D = \frac{d}{dt}$.

$Then\ given\ equation\ reduces\ to\ \ 4D(D-1)y - 12Dy + 16y = 8e^{2t}$.

$\therefore \ (D^2 - 4D + 4)y = 2e^{2t}$. $\quad \therefore$ AE is $m^2 - 4m + 4 = 0$. $\therefore \ (m-2)^2 = 0$.

$\therefore \ (m-2)(m-2) = 0. \ \therefore \ m = 2,\ 2. \quad \therefore \ CF = (c_1 + tc_2)e^{2t}$.

$PI = \frac{2e^{2t}}{(D^2-4D+4)} = \frac{2te^{2t}}{2D-4} = \frac{2t^2e^{2t}}{2} = t^2 e^{2t}$.

$\therefore$ The general solution is given by $y = CF + PI$. $\ \therefore \ y = (c_1 + c_2 t) + t^2 e^{2t}$.

$\therefore \quad Put\ t = \log(1+2x)\ \ or\ 1 + 2x = e^t$.

$\therefore \quad y = [c_1 + c_2 \log(2x+1)](2x+1)^2 + [\log(2x+1)]^2(2x+1)^2.$

14. Solve $(1+x)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = \sin 2[\log(1+x)]$.

Solution:

Given $(1+x^2)y'' + (1+x)y' + y = \sin2[\log(1+x)].$

$Put\ t = \log(1+x)\ \ or\ \ 1+\ x = e^t.$

$\therefore \ (1+x)y' = 1.Dy \quad ,(1+x)^2y'' = 1^2 .\ D(D-1)y.$ Where $D = \frac{d}{dt}$

$Then\ given\ equation\ reduces\ to \quad D(D-1)y + Dy + y = sin2t.$

$\therefore \quad (D^2+1))y = sin2t. \quad \therefore$ AE is $m^2 + 1 = 0. \quad \therefore \ m = \pm\, i.$

$\therefore \quad CF = c_1 \cos t + c_2 sint\ .$

$PI = \frac{sin2t}{D^2+1} = \frac{sin2t}{-3}.\ \ \therefore$ The general solution is given by $y = CF + PI.$

$\therefore \quad y = c_1 \cos t + c_2 \sin t - \frac{sin2t}{3}\ . \quad Put\ t = \log(1+x)\ or 1+\ x = e^t.$

$\therefore \quad y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \frac{\sin[2\log(1+x)]}{3}.$

15. Solve $(2x+3)^2 \frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x$.

Solution:

Given $(2x+3)^2 \frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x.$

$Put\ t = \log(2x+3)\ \ or\ \ 3+\ 2x = e^t.$

$\therefore \ (2x+3)y' = 2.Dy,\ (2x+3)^2y'' = 2^2 .\ D(D-1)y.$ Where $D = \frac{d}{dt}$.

$Then\ given\ equation\ reduces\ to \quad 4D(D-1)y - 2Dy - 12y = 3(e^t - 3).$

$\therefore \ (2D^2 - 3D - 6)y = \frac{3(e^t-3)}{2}. \quad \therefore \ \left(D^2 - \frac{3}{2}D - 3\right)y = \frac{3(e^t-3)}{4}.$

$\therefore$ AE is $m^2 - \frac{3}{2}m - 3 = 0.\ \therefore m = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{\frac{3}{2}\pm\sqrt{\frac{9}{4}+12}}{2} = \frac{\frac{3}{2}\pm\sqrt{\frac{57}{4}}}{2} = \frac{3\pm\sqrt{57}}{4}.$

$\therefore \quad CF = c_1 e^{\left(\frac{3+\sqrt{57}}{4}\right)t} + c_2 e^{\left(\frac{3-\sqrt{57}}{4}\right)t}.$

$PI = \frac{3(e^t-3)}{4(D^2-\frac{3}{2}D-3)} = \frac{3}{4}\left(\frac{e^t}{(D^2-\frac{3}{2}D-3)} - \frac{3}{(D^2-\frac{3}{2}D-3)}\right) = \frac{3}{4}\left(\frac{2e^t}{-7} - \frac{3}{-3}\right) = \frac{3}{4}\left(\frac{2e^t}{-7} + 1\right).$

$\therefore$ The general solution is given by $y = CF + PI.$

$\therefore \quad y = c_1 e^{\left(\frac{3+\sqrt{57}}{4}\right)t} + c_2 e^{\left(\frac{3-\sqrt{57}}{4}\right)t} + \frac{3}{4}\left(\frac{2e^t}{-7} + 1\right).\ \ Put\ t = \log(2x+3)\ or\ 2x+3 = e^t.$

$$\therefore \quad y = c_1(2x+3)^{\left(\frac{3+\sqrt{57}}{4}\right)} + c_2(2x+3)^{\left(\frac{3-\sqrt{57}}{4}\right)} + \frac{3}{4}\left(\frac{2(2x+3)}{-7} + 1\right).$$

Home work

1. Solve $\boldsymbol{(x+1)^2 y'' + (x+1)y' + y = 4\cos[\log(x+1)]}$