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NAGARJUNA
COLLEGE OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

ADVANCED CALCULUS AND NUMERICAL METHODS

(COURSE CODE 22MATS21/22MATE21/22MATC21)

CLASS NOTES FOR II SEM B.E.

MODULE 3

NUMERICAL METHODS-2

MODULE 3.

NUMERICAL METHODS-2

SYLLABUS:
Numerical Solution of Ordinary Differential Equations (ODE's):
Numerical solution of ordinary differential equations of first order and first degree: Taylor's series method, Modified Euler's method, Runge-Kutta method of fourth-order, and Milne's predictor-corrector formula (No derivations of formulae). Problems.

Numerical solutions of ordinary differential equations of first order and first degree:

Many differential equations cannot be solved analytically. For practical purposes in engineering, a numeric approximation to the solution is often sufficient. The algorithms used here are known as numerical methods.

Numerical methods for ordinary differential equations are the methods used to find numerical approximations to the solutions of ordinary differential equations under given conditions which are also known as initial value problems.

Taylor's Series Method

Consider $\frac{dy}{dx} = f(x,y)\ with\ y(x_0) = y_0$. The solution of this equation is given by the

Taylor's series expansion of $y(x)$ about the point x_0 in the form

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \ldots\ldots\ldots$$

Problems:

1. Find by Taylor's series method the values of y at $x = 0.1\ and\ x = 0.2$ to 5 places of decimals from $\frac{dy}{dx} = x^2y - 1\ ,\ y(0) = 1$.

Solution:

Given $\frac{dy}{dx} = x^2y - 1,\ with\ y(0) = 1$. i.e., $\frac{dy}{dx} = f(x,y)\ with\ y(x_0) = y_0$.

$\therefore\ f(x,y) = x^2y - 1.\ \ x_0 = 0,\ and\ y_0 = 1.$

Taylor's series expansion of $y(x)$ about the point x_0 is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \frac{(x-x_0)^4}{4!}y^{iv}(x_0) + \ldots\ldots\ldots$$

$$\therefore\ y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \cdots \ldots\ldots(1)$$

Now, given $y' = x^2y - 1$. $\therefore$ $y'(0) = -1$.

$y'' = 2xy + x^2y'$. $\therefore$ $y''(0) = 0$.

$y''' = 2y + 4xy' + x^2y''$. $\therefore$ $y'''(0) = 2$.

$y^{iv} = 6y' + 6xy'' + x^2y'''$. $\therefore$ $y^{iv}(0) = -6$.

Substituting in (1) we get,

$$y(x) = 1 - x + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \cdots$$

$$\therefore \quad y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots \qquad \ldots\ldots\ldots (2)$$

Put x = 0.1 and x = 0.2 in (2).

$\therefore$ $y(0.1) = 0.90033,$ $y(0.2) = 0.80227.$

2. Employ Taylor's method to obtain approximate value of y at $x = 0.2$ for the differential equation $\frac{dy}{dx} = 2y + 3e^x, y(0) = 0$.

Solution:

Given $\frac{dy}{dx} = 2y + 3e^x$ $with$ $y(0) = 0$. i.e., $\frac{dy}{dx} = f(x,y)\ with\ y(x_0) = y_0$.

$\therefore$ $f(x,y) = 2y + 3e^x$. $x_0 = 0$, $and\ y_0 = 0$.

Taylor's series expansion of $y(x)$ about the point x_0 is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \ldots\ldots\ldots\ldots$$

$$\therefore \quad y(x) = y(0) + x\,y'(x_0) + \frac{x^2}{2!}y''(x_0) + \frac{x^3}{3!}y'''(x_0) + \ldots\ldots\ldots \; (1)$$

Now, given $y' = 2y + 3e^x$ $\therefore$ $y'(0) = 3$

$y'' = 2y' + 3e^x$ $\therefore$ $y''(0) = 9$

$y''' = 2y'' + 3e^x$ $\therefore$ $y'''(0) = 21$

Substituting in (1) we get,

$$y(x) = 0 + x(3) + \frac{x^2}{2!}(9) + \frac{x^3}{3!}(21) + \cdots$$

$$\therefore \quad y(x) = 0 + 3x + \frac{9x^2}{2!} + \frac{21x^3}{3!} + \ldots\ldots\ldots\ldots$$

Put x = 0.2.

$$\therefore \quad y(0.2) = 0 + 9(0.2) + \frac{9(0.04)}{2} + \frac{21(0.008)}{6} = 0.808.$$

3. Using Taylor's series method, solve $y' = x + y^2$, $y(0) = 1$ at $x = 0.1$, 0.2 considering upto fourth degree term.

Solution:

Given $\frac{dy}{dx} = x + y^2$ with $y(0) = 1$. i.e., $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

$\therefore$ $f(x, y) = x + y^2$. $x_0 = 0$, and $y_0 = 1$.

Taylor's series expansion of $y(x)$ about the point x_0 is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \frac{(x-x_0)^4}{4!}y^{iv}(x_0) + \ldots\ldots..$$

$$\therefore \quad y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots\ldots\ldots(1)$$

Now, given $y' = x + y^2$. $\therefore$ $y'(0) = 0 + [y(0)]^2 = 0 + 1 = 1$

$y'' = 1 + 2yy'$. $\therefore$ $y''(0) = 1 + 2y(0)y'(0) = 1 + 2 = 3$

$y''' = 2[yy'' + y'^2]$. $\therefore$ $y'''(0) = 2[(1)(3) + 1] = 8$

$y^{iv} = 2[yy''' + yy'' + 2yy''] = 2[yy''' + 3yy''']$. $\therefore$ $y^{iv}(0) = 2(1)(8) + 6(1)(3) = 34$

Substituting in (1), we get,

$$y(x) = 1 + x + \frac{x^2}{2!}(3) + \frac{x^3}{3!}(8) + \frac{x^4}{4!}(34) + \cdots$$

$$\therefore \quad y(x) = 1 + x + \frac{3x^2}{2} + \frac{8x^3}{6} + 34\frac{x^4}{24} + \cdots$$

Put x = 0.1 and x = 0.2. $\therefore$ $y(0.1) = 1.1165$ and $y(0.2) = 1.2723$

4. Solve by Taylor's Series method the equation $\frac{dy}{dx} = \log\,(xy)$ for $y(1.1)$ and $y(1.2)$, given $y(1) = 2$.

Solution:

Given $\frac{dy}{dx} = \log\,(xy)$ with $y(1) = 2$. i.e., $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

$\therefore$ $f(x, y) = \log\,(xy)$. $x_0 = 1$ and $y_0 = 2$.

Taylor's series expansion of $y(x)$ about the point x_0 is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \ldots\ldots\ldots\ldots$$

$$\therefore \quad y(x) = y(1) + (x - 1)y'(1) + \frac{(x-1)^2}{2!}y''(1) + \frac{(x-1)^3}{3!}y'''(1)\ldots\ldots\ldots(1)$$

Now, given $y' = log\,(x) + log\,(y)$ $\therefore$ $y'(1) = log\,2$

$y'' = \frac{1}{x} + \frac{1}{y}y'$. $\therefore$ $y''(1) = 1 + \frac{1}{2}\log 2$

$y''' = -\frac{1}{x^2} + \frac{1}{y}.y'' + y'\left(-\frac{1}{y^2}\right)y'$. $\therefore$ $y'''(1) = -1 + \frac{1}{2}\left(1 + \frac{1}{2}\log 2\right) - \frac{1}{4}(\log 2)^2$.

$\therefore$ $y'''(1) = -1 + \frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2 = -\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2$.

Substituting in (1), we get,

$y(x) = 2 + (x-1)\, log\, 2 + \frac{(x-1)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(x-1)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right] + \cdots$

Put x = 1.1 and x = 1.2.

$\therefore$ $y(1.1) = 2 + (0.1)\, log\, 2 + \frac{(0.1)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(0.1)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right] + \cdots$

$\therefore$ $y(1.1) = 2.0961$. (Use natural logarithm) and

$$y(1.2) = 2 + (0.2)\, log\, 2 + \frac{(0.2)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(0.2)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right] + \cdots$$

$\therefore$ $y(1.2) = 2.1647$.

HOMEWORK:

1. **Find y at $x = 1.02$ correct to 5 decimal places given $dy = (xy - 1)dx$ *and* $y = 2$ *at* $x = 1$ applying Taylor's Series method.   Ans: $y(1.02) = 2.0206$.**
2. **Use Taylor's Series method to find an approximate value of y when $x = 0.1$, if $\frac{dy}{dx} = x - y^2$ and $x_0 = 0, y_0 = 1$ initially.   Ans: 0.9138.**

Modified Euler's method:

Consider initial value problems $\frac{dy}{dx} = f(x,y)\ with\ y(x_0) = y_0$. We determine the solution of this problem at the point $x_n = x_0 + nh$. Where h is the step length.

To find $y_1 = y(x_1)$, where $x_1 = x_0 + h$, we first find the initial approximation to y_1 by using Euler's formula $y_1^{(0)} = y_0 + hf(x_0, y_0)$. Next we find a better approximation by using the Modified Euler's Formula $y_n^{(p)} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f\left(x_n, y_n^{(p-1)}\right)\right]$ to the desired degree of accuracy as follows:

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$ and so on.

To find $y_2 = y(x_2)$, where $x_2 = x_1 + h$, we first find the initial approximation to y_2 by using Euler's formula $y_2^{(0)} = y_1 + hf(x_1, y_1)$. Next we find a better approximation by using the Modified Euler's Formula $y_n^{(p)} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f\left(x_n, y_n^{(p-1)}\right)\right]$ to the desired degree of accuracy as follows:

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right]$$

$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(2)}\right)\right]$ and so on.

Problems:

1. Using Modified Euler's method, find an approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x + y$ and $y = 1$ $at\ x = 0$. Take h = 0.2, perform 4 iteration.

Solution:

Given $\frac{dy}{dx} = x + y$ $with$ $y(0) = 1$. i.e., $\frac{dy}{dx} = f(x, y)\ with\ y(x_0) = y_0$.

$\therefore$ $f(x, y) = x + y$, $x_0 = 0$ $and\ y_0 = 1$. Take h = 0.2. $\therefore$ $x_1 = x_0 + h$ = 0 + 0.2 = 0.2.

And $f(x_0, y_0) = 0 + 1 = 1$.

By Euler's formula, we have, $y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + (0.2)1 = 1.2$.

$\therefore$ $f\left(x_1, y_1^{(0)}\right)$ = 0.2 + 1.2 = 1.4

Using Modified Euler's formula, we have,

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right] = 1 + \frac{0.2}{2}[1 + 1.14] = 1.24$.

$\therefore$ $f\left(x_1, y_1^{(1)}\right)$ = 0.2 + 1.24 = 1.44.

$\therefore\ y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right] = 1 + \frac{0.2}{2}[1 + 1.44] = 1.244.$

$\therefore\ f\left(x_1, y_1^{(2)}\right) = 0.2 + 1.244 = 1.444.$

$\therefore\ y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right] = 1 + \frac{0.2}{2}[1 + 1.444] = 1.2444.$

$\therefore\ f\left(x_1, y_1^{(3)}\right) = 0.2 + 1.2444 = 1.4444.$

$\therefore\ y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(3)}\right)\right] = 1 + \frac{0.2}{2}[1 + 1.4444] = 1.24444.$

$\therefore\ y(0.2) = 1.24444.$

2. Using Modified Euler's Method find $y(0.2)$ correct to 4 decimal places given that $\frac{dy}{dx} = x - y^2,\ \ y(0) = 1.$ Taking h = 0.1, use modified formula twice in each stages.

Solution:

Given $\frac{dy}{dx} = x - y^2\ \ with\ \ y(0) = 1.$ i.e., $\frac{dy}{dx} = f(x, y)\ with\ y(x_0) = y_0.$

$\therefore\ f(x, y) = x - y^2,\ \ x_0 = 0\ \ and\ y_0 = 1.$ Take h = 0.1. $\therefore\ x_1 = x_0 + h = 0 + 0.1 = 0.1$

and $x_2 = x_1 + h = 0.1 + 0.1 = 0.2.$ Hence we find the value $y(0.2)$ in two stages.

We shall first compute $y(0.1)$ and use this value to compute $y(0.2)$

First stage :

Now, $f(x_0, y_0) = 0 - 1^2 = -1\ \ and\ \ x_1 = 0.1$

By Euler's formula, $y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + (0.1)(-1) = 0.9.$

$\therefore\ f\left(x_1, y_1^{(0)}\right) = 0.1 - (0.9)^2 = -0.71.$

By Modified Euler's formula, $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$, we get,

$y_1^{(1)} = 1 + \frac{0.1}{2}[-1 - 0.71] = 0.9145$

$\therefore\ f\left(x_1, y_1^{(1)}\right) = 0.1 - (0.9145)^2 = -0.7363.$

$\therefore\ y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right] = 1 + \frac{0.1}{2}[-1 - 0.7363] = 0.9132.$

$\therefore\ y(0.1) = 0.9132.$

Second stage:

Now take $x_1 = 0.1,\ y_1 = 0.9132\ \ and\ \ f(x, y) = x - y^2.$

$f(x_1, y_1) = 0.1 - (0.9132)^2 = -0.7339$ and $x_2 = 0.2.$

By Euler's formula, $y_2^{(0)} = y_1 + hf(x_1, y_1) = 0.9132 + (0.1)(-0.7339) = 0.8398$.

$\therefore \; f\left(x_2, y_2^{(0)}\right) = 0.2-(0.8398)^2 = -0.5053$.

By Modified Euler's formula, $y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$, we get,

$y_2^{(1)} = 0.9132 + \frac{0.1}{2}[-0.7339 - 0.5053] = 0.8512$.

$\therefore \; f\left(x_2, y_2^{(1)}\right) = 0.2-(0.8512)^2 = -0.5245$.

$\therefore \; y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right]$

$\therefore \; y_1^{(2)} = 0.9132 + \frac{0.1}{2}[-0.7339 - 0.5245] = 0.8503$

$\therefore \; y(0.2) = 0.8503$.

3. Use Modified Euler's Method to solve $\frac{dy}{dx} = \log(x + y)$, $y(0) = 2$ at $x = 0.2$ and $x = 0.4$ with $h = 0.2$. (Use modified formula twice)

Solution:

Given $\frac{dy}{dx} = \log\,(x + y)$ with $y(0) = 2$. i.e., $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

$\therefore$ $f(x, y) = \log\,(x + y)$, $x_0 = 0$ and $y_0 = 2$. Take $h = 0.2$. $\therefore$ $x_1 = x_0 + h = 0 + 0.2 = 0.2$.

and $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$. Hence we find the value $y(0.4)$ in two stages.

We shall first compute $y(0.2)$ and use this value to compute $y(0.4)$.

First stage :

Now, $f(x_0, y_0) = \log\,(0 + 2) = 0.6931$. and $x_1 = 0.2$

By Euler's formula, $y_1^{(0)} = y_0 + hf(x_0, y_0) = 2 + (0.2)(0.6931) = 2.1286$.

$\therefore \; f\left(x_1, y_1^{(0)}\right) = \log(0.2 + 2.1286) = 0.8453$.

By Modified Euler's formula, $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$, we get,

$y_1^{(1)} = 2 + \frac{0.2}{2}[0.6931 + 0.8453] = 2.1538$.

$\therefore \; f\left(x_1, y_1^{(1)}\right) = \log(0.2 + 2.1538) = 0.8560$.

$\therefore \; y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right] = 2 + \frac{0.2}{2}[0.6931 + 0.8560] = 2.1549$.

$\therefore \quad y(0.2) = 2.1549.$

Second stage:

Now take $x_1 = 0.2,\; y_1 = 2.1549 \; and \; f(x,y) = \log\,(x + y).$

$f(x_1, y_1) = log(0.2 + 2.1549) = 0.8565$ and $x_2 = 0.4.$

By Euler's formula, $y_2^{(0)} = y_1 + hf(x_1, y_1) = 2.1549 + (0.2)(0.8565) = 2.3262.$

$\therefore \quad f\left(x_2, y_2^{(0)}\right) = \log(0.4 + 2.3262) = 1.0029.$

By Modified Euler's formula, $y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$, we get,

$y_2^{(1)} = 2.1549 + \frac{0.2}{2}[0.8565 + 1.0029] = 2.3435.$

$\therefore \quad f\left(x_2, y_2^{(1)}\right) = \log(0.4 + 2.3435) = 1.0092.$

$\therefore \quad y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right]$

$\therefore \quad y_2^{(2)} = 2.1549 + \frac{0.2}{2}[0.8565 + 1.0092] = 2.3415$

$\therefore \quad y(0.4) = 2.3415.$

4. Use Modified Euler's Method to solve $\frac{dy}{dx} = x + |\sqrt{y}|$ in the range $0 \le x \le 0.4$ by taking h= 0.2 given that $y = 1 \; at \; x = 0$ initially. Use modified formula twice.

Solution:

Given $\frac{dy}{dx} = x + |\sqrt{y}|$ with $y(0) = 1.$ i.e., $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0.$

$\therefore \quad f(x,y) = x + |\sqrt{y}|,\; x_0 = 0$ and $y_0 = 1.$ Take $h = 0.2.$ $\therefore\; x_1 = x_0 + h = 0 + 0.2 = 0.2.$

and $x_2 = x_1 + h = 0.2 + 0.2 = 0.4.$ Hence we find the value $y(0.4)$ in two stages.

We shall first compute $y(0.2)$ and use this value to compute $y(0.4)$.

First stage :

Now, $f(x_0, y_0) = 0 + \sqrt{1} = 1.$ and $x_1 = 0.2.$

By Euler's formula, $y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + (0.2)1 = 1.2.$

$\therefore \quad f\left(x_1, y_1^{(0)}\right) = 0.2 + \sqrt{1.2} = 1.2954.$

By Modified Euler's formula, $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$, we get,

$y_1^{(1)} = 1 + \frac{0.2}{2}[1 + 1.2954] = 1.2295.$ $\qquad f(x,y) = x + |\sqrt{y}|$

$\therefore \; f\left(x_1, y_1^{(1)}\right) = 0.2 + \sqrt{1.2295} = 1.3088.$

$\therefore \; y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right] = 1 + \frac{0.2}{2}[1 + 1.3382] = 1.2309.$

$\therefore \; y(0.2) = 1.2309.$

Second stage:

Now take $x_1 = 0.2, \; y_1 = 1.2309 \; and \; f(x, y) = x + |\sqrt{y}|.$

$f(x_1, y_1) = 0.2 + \sqrt{1.2309} = 1.3095$ and $x_2 = 0.4.$

By Euler's formula, $y_2^{(0)} = y_1 + hf(x_1, y_1) = 1.2309 + (0.2)(1.3095) = 1.4928.$

$\therefore \; f\left(x_2, y_2^{(0)}\right) = 0.4 + \sqrt{1.4928} = 1.6218.$

By Modified Euler's formula, $y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$, we get,

$y_2^{(1)} = 1.2309 + \frac{0.2}{2}[1.3095 + 1.6218] = 1.5240.$

$\therefore \; f\left(x_2, y_2^{(1)}\right) = 0.4 + \sqrt{1.5240} = 1.6345.$

$\therefore \; y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right] = 1.2309 + \frac{0.2}{2}[1.3095 + 1.6345] = 1.5253$

$\therefore \; y(0.4) = 1.5253.$

HOMEWORK:

1. Using Modified Euler's Method, find $y(0.2)$ and $y(0.4)$ given $y' = y + e^x, y(0) = 0.$

(take h = 0.2). Ans: y(0.2)=0.2468 and y(0.4)= 0.6030

2. Using Modified Euler's Method, find $y(0.2)$ and $y(0.4)$ given $\frac{dy}{dx} = x + \sin x$ with $y(0) = 1$ taking h = 0.2. Perform 3 iteration at each step. (x in radians).

Ans: y(0.2)=1.1973 and y(0.2)= 1.4766.

Runge-Kutta method of fourth order:

Consider the initial value problem $\frac{dy}{dx} = f(x, y) \; with \; y(x_0) = y_0$. We determine the solution of this problem at the point $x_n = x_0 + nh$. Where h is the step length.

To find $y_1 = y(x_1)$, where $x_1 = x_0 + h$, we first compute k_1, k_2, k_3, k_4 by using the following formulae,

$k_1 = hf(x_0,\ y_0)$

$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$

$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$

$k_4 = hf(x_0 + h,\ y_0 + k_3)$

The required $y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.

To find $y_2 = y(x_2)$, where $x_2 = x_1 + h$, we first compute $k_1,\ k_2,\ k_3,\ k_4$ by using the following formulae,

$k_1 = hf(x_1,\ y_1)$

$k_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right)$

$k_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right)$

$k_4 = hf(x_1 + h,\ y_1 + k_3)$

The required $y(x_2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$. And so on.

PROBLEMS:

1. Apply Runge-Kutta method of fourth order, to find an approximate value of y when x = 0.2, given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$.

Solution:

Given $\frac{dy}{dx} = x + y$ with $y(0) = 1$. i.e., $\frac{dy}{dx} = f(x,\ y)$ with $y(x_0) = y_0$.

$\therefore$ $f(x,\ y) = x + y$, $x_0 = 0$ and $y_0 = 1$. Take $h = 0.2$. $\therefore$ $x_1 = x_0 + h = 0 + 0.2 = 0.2$.

Using Runge-Kutta method of fourth order, we find,

$k_1 = hf(x_0,\ y_0) = (0.2)(0 + 1) = 0.2$

$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = (0.2)f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.2}{2}\right) = (0.2)f(0.1, 1.1)$

$\therefore$ $k_2 = (0.2)(0.1 + 1.1) = 0.24$

$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = (0.2)f(0.1, 1.12) = (0.2)(0.1 + 1.12) = 0.244$

$k_4 = hf(x_0 + h,\ y_0 + k_3) = (0.2)f(0.2, 1.244) = (0.2)(0.2 + 1.244) = 0.2888$

$y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ =1+$\frac{1}{6}$[0.2 + 2(0.24) + 2(0.244) + 0.2888]=1.2428.

$\therefore\ y(0.2) = 1.2428.$

2. Apply Runge-Kutta method of fourth order, to find an approximate value of y when x = 0.2, in steps of 0.1 if $\frac{dy}{dx} = x + y^2$ given that $y = 1$ *when* $x = 0$.

Solution:

Given $\frac{dy}{dx} = x + y^2$ with $y(0) = 1$. i.e., $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

$\therefore$ $f(x, y) = x + y^2$, $x_0 = 0$ and $y_0 = 1$. Take h = 0.1. $\therefore$ $x_1 = x_0 + h$ = 0 + 0.1 = 0.1

and $x_2 = x_1 + h$ = 0.1 + 0.1 = 0.2. Hence by using Runge-Kutta method of fourth order, we find the value $y(0.2)$ in two steps. We shall first compute $y(0.1)$ and use this value to compute $y(0.2)$

First Step:

To find $y(0.1)$, *take* $x_0 = 0$, $y_0 = 1$, h = 0.1, $x_1 = 0.1$ and $f(x, y) = x + y^2$.

$\therefore\ k_1 = hf(x_0, y_0)$ =(0.1) $f(0, 1)$= (0.1)(0 + 1^2) = 0.1.

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f(0.05, 1.1) = (0.1)(0.05 + (1.1)^2) = 0.1152$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f(0.05, 1.1152) = (0.1)(0.05 + (1.1151)^2) = 0.1168$

$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)f(0.1, 1.1168) = (0.1)(0.1 + (1.1168)^2) = 0.1347$

$\therefore\ y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ = 1 + $\frac{1}{6}$[0.1 + 2(0.1152) + 2(0.1168) + 0.1347]

$\therefore\ y(0.1)$ = 1.1165

Second Step:

To find $y(0.2)$, *take* $x_1 = 0.1$, $y_1 = 1.1165$, h = 0.1, $x_2 = 0.2$ and $f(x, y) = x + y^2$.

$\therefore\ k_1 = hf(x_1, y_1) = (0.1)f(0.1, 1.1165) = (0.1)[0.1 + (1.1165)^2]$ = 0.1347

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)f(0.15, 1.1838) = 0.1551$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1)f(0.15, 1.194) = 0.1576$

$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)f(0.2, 1.1576) = 0.1823$

$\therefore\ y(x_2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$

$\therefore\ y(x_2) = 1 + \frac{1}{6}[0.1347 + 2(0.1551) + 2(0.1596) + 0.1823] = 1.2736$

$\therefore\ y(0.2)$ = 1.2736.

3. Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with y(0)=1 at x= 0.2, 0.4.

Solution:

Given $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0) = 1$. i.e., $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

$\therefore$ $f(x, y) = \frac{y^2-x^2}{y^2+x^2}$, $x_0 = 0$ and $y_0 = 1$. Take $h = 0.2$. $\therefore$ $x_1 = x_0 + h = 0 + 0.2 = 0.2$.

and $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$. Hence by using Runge-Kutta method of fourth order, we find, the value $y(0.4)$ in two steps. We shall first compute $y(0.2)$ and use this value to compute $y(0.4)$.

First Step:

To find $y(0.2)$, $take\ x_0 = 0,\ y_0 = 1$, h = 0.2, $x_1 = 0.2$ and $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$.

$\therefore\ k_1 = hf(x_0, y_0)$ =(0.2) $f(0,1)$= 0.2

$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = (0.2)f(0.1, 1.1) = 0.1967$

$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = (0.2)f(0.1, 1.0984) = 0.1967$

$k_4 = hf(x_0 + h,\ y_0 + k_3) = (0.2)f(0.2, 1.1967) = 0.1891$

$\therefore$ $y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{1}{6}[0.2 + 2(0.1967) + 2(0.1967) + 0.1891]$

$\therefore$ $y(0.2)$=1.1960.

Second Step:

To find $y(0.2)$, $take\ x_1 = 0.1,\ y_1 = 1.1960$, h = 0.1, $x_2 = 0.2$ and $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$

$\therefore\ k_1 = hf(x_1, y_1) = 0.1891$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.2)f(0.3,1.2906) = 0.1795$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.2)f(0.3,1.2858) = 0.1793$

$k_4 = hf(x_1 + h, y_1 + k_3) = (0.2)f(0.4,1.3753) = 0.1688$

$\therefore$ $y(x_2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore$ $y(x_2) = 1.1165 + \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] = 1.3752.$

$\therefore$ $y(0.4) = 1.3752.$

4. Apply Runge-Kutta method of fourth order, solve for y at $x = 1.2$ from $\frac{dy}{dx} = \frac{2xy+e^x}{x^2+xe^x}$ given $y_0 = 0$ and $x_0 = 1$.

Solution:

Given $\frac{dy}{dx} = \frac{2xy+e^x}{x^2+xe^x}$ with $y(1) = 0$. i.e., $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

$\therefore$ $f(x, y) = \frac{2xy+e^x}{x^2+xe^x}$, $x_0 = 1$ and $y_0 = 0$. Take h = 0.2. $\therefore$ $x_1 = x_0 + h = 1 + 0.2 = 1.2$.

Using Runge-Kutta method of fourth order, we find,

$k_1 = hf(x_0, y_0) = (0.2)f(1, 0) = 0.1462$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)f(1.1, 0.0731) = 0.1402$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)f(1.1, 0.0701) = 0.1399$

$k_4 = hf(x_0 + h, y_0 + k_3) = (0.2)f(1.2, 0.1399) = 0.1348$

$\therefore$ $y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y(1.2) = 0 + \frac{1}{6}[0.1462 + 2(0.1402) + 2(0.1399) + 0.1348] = 0.1402$.

$\therefore$ $y(1.2) = 0.1402$.

HOMEWORK:

1. Use Runge-Kutta method of fourth order to find $y(0.2)$ given that $\frac{dy}{dx} = 3x + \frac{1}{2}y$, $y(0) = 1$, taking $h = 0.1$. Ans: $y(0.2) = 1.1672$

2. Using Runge-Kutta method of order four, find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$. Take $h = 0.2$. Ans: $y(0.2) = 1.1678$.

MILNE'S M ETHOD:

Consider the initial value problem $\frac{dy}{dx} = f(x, y)$ *with* $y(x_0) = y_0, y(x_1) = y_1, y(x_2) = y_2$, $y(x_3) = y_3$, where $x_1 = x_0 + h$, $x_2 = x_1 + h$, $x_3 = x_2 + h$ and h is the step length.

We determine the solution $y(x_4) = y_4$ of this problem at the point $x_4 = x_3 + h$, as follows.

We first predict the value of y_4 by using the formula,

$y_4^{(p)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$. Where $y_1' = f(x_1, y_1)$, $y_2' = f(x_2, y_2)$ and $y_3' = f(x_3, y_3)$.

This formula is called the Milne's predictor formula.

Further we correct the value of y_4 by using the formula,

$$y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4'). \quad \text{Where } y_4' = f\left(x_4, y_4^{(p)}\right)$$

This formula is called the Milne's corrector formula.
This method is known as Milne's predictor and corrector method.
This method is also known as multi-step method. Previous methods are called the single step methods.

PROBLEMS:

1. Given $y' = x - y^2$, using the data $y(0) = 0,\ y(0.2) = 0.02,\ y(0.4) = 0.0795,$ $y(0.6) = 0.1762$. Compute y at x = 0.8 by applying Milne's method.

Solution:

We prepare the following table using the given data: Here h = 0.2.

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	
$x_1 = 0.2$	$y_1 = 0.02$	$y'_1 = 0.2 - (0.02)^2 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y'_2 = 0.4 - (0.0795)^2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$y'_3 = 0.6 - (0.1762)^2 = 0.5689$
$x_4 = 0.8$	$y_4 = ?$	

Using Milne's predictor formula, $y_4^{(p)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$, we get,

$$y_4^{(p)} = 0 + \frac{4(0.2)}{3}[2(0.1996) - 0.3937 + 2(0.5689)] = 0.3049.$$

$$\therefore \quad y_4' = f\left(x_4, y_4^{(p)}\right) = 0.8 - (0.3049)^2 = 0.707$$

Using Milne's corrector formula, $y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$, we get,

$$y_4^{(c)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.707] = 0.3046$$

$$\therefore \quad y_4' = f\left(x_4, y_4^{(p)}\right) = 0.8 - (0.3049)^2 = 0.7072.$$

Substituting this value of y_4' again in the corrector formula,

$$y_4^{(c)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.7072] = 0.3046$$

$$\therefore \quad y_4 = y(0.8) = 0.3046.$$

2. If $\frac{dy}{dx} = 2e^x - y,\ y(0) = 2,\ y(0.1) = 2.010,\ y(0.2) = 2.040$ *and* $y\ (0.3) = 2.090$ find y(0.4) correct to four decimal places by using Milne's method.

Solution:

We prepare the following table using the given data: Here h = 0.1.

x	y	$\boldsymbol{y'} = 2e^{x} - y$
$x_0 = 0$	$y_0 = 2$	
$x_1 = 0.1$	$y_1 = 2.010$	$y'_1 = 2e^{0.1} - 2.010 = 0.2003$
$x_2 = 0.2$	$y_2 = 2.040$	$y'_2 = 2e^{0.2} - 2.040 = 0.4028$
$x_3 = 0.3$	$y_3 = 2.090$	$y'_3 = 2e^{0.3} - 2.090 = 0.6097$
$x_4 = 0.4$	$y_4 = ?$	

Using Milne's predictor formula, $y_4^{(p)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$, we get,

$$y_4^{(p)} = 2 + \frac{4(0.1)}{3}[2(0.2003) - (0.4028) + 2(0.6097)] = 2.1623.$$

$$\therefore \quad y_4' = f\left(x_4, y_4^{(p)}\right) = 2e^{0.4} - 2.1623 = 0.8213.$$

Using Milne's corrector formula, $y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$, we get,

$$y_4^{(c)} = 2.040 + \frac{0.1}{3}[0.4028 + 4(0.6097) + 0.8213] = 2.1621.$$

$$\therefore \quad y_4' = 2e^{0.4} - 2.1621 = 0.8215.$$

Substituting this value of y_4' again in the corrector formula

$$y_4^{(c)} = 2.040 + \frac{0.1}{3}[0.4028 + 4(0.6097) + 0.8215] = 2.1621. \quad \therefore \quad y_4 = y(0.4) = 2.1621.$$

3. Apply Milne's method to compute y(1.4) correct to four decimal places given $\frac{dy}{dx} = x^2 + \frac{y}{2}$ and the following data $y(1) = 2$, $y(1.1) = 2.2156$, $y(1.2) = 2.4649$, $y(1.3) = 2.7514$.

Solution:

We prepare the following table using the given data: Here h = 0.1.

x	y	$\boldsymbol{y'} = x^2 + \frac{y}{2}$
$x_0 = 1$	$y_0 = 2$	
$x_1 = 1.1$	$y_1 = 2.2156$	$y'_1 = (1.1)^2 + \frac{2.2156}{2} = 2.3178$
$x_2 = 1.2$	$y_2 = 2.4649$	$y'_2 = (1.2)^2 + \frac{2.4649}{2} = 2.6725$
$x_3 = 1.3$	$y_3 = 2.7514$	$y'_3 = (1.3)^2 + \frac{2.7514}{2} = 3.0657$
$x_4 = 1.4$	$y_4 = ?$	

Using Milne's predictor formula, $y_4^{(p)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$, we get,

$y_4^{(p)} = 2 + \frac{4(0.1)}{3}[2(2.3178)\ - (2.6725) + 2(3.0657)] = 3.0793$.

$\therefore \quad y_4' = f\left(x_4, y_4^{(p)}\right) = (1.4)^2 + \frac{3.0793}{2} = 3.4996.$

Using Milne's corrector formula, $y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$, we get,

$y_4^{(c)} = 2 + \frac{(0.1)}{3}[2.6725 + 4(3.0657) + 3.4996] = 3.0794.$

Now $y_4' = f\left(x_4, y_4^{(p)}\right) = (1.4)^2 + \frac{3.07945}{2} = 3.4997$

Substituting this value of y_4' again in the corrector formula,

$y_4^{(c)} = 2 + \frac{(0.1)}{3}[2.6725 + 4(3.0657) + 3.4996] = 3.0794. \quad \therefore \quad y_4 = y(1.4) = 3.0794.$

HOMEWORK:

1. Given $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$, $y(0.1) = 1.1169$, $y(0.2) = 1.2773$, $y(0.3) = 1.5049$. Find $y(0.4)$ correct to three decimal places, using the Milne's predictor-corrector method. Apply the corrector formula thrice. Ans: $y(0.4) = 1.840$

2. Using the Milne's method, solve the differential equation $\frac{dy}{dx} = \frac{2y}{x}, x \neq 0$, at the point $x = 2$. Given that $y(1) = 2$, $y(1.25) = 3.13$, $y(1.5) = 4.5$, and $y(1.75) = 6.13$. Apply the corrector formula twice. Ans: $y(2) = 8.0021$