NAGARJUNA COLLEGE OF ENGINEERING AND TECHNOLOGY
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NAGARJUNA
COLLEGE OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

ADVANCED CALCULUS AND NUMERICAL METHODS

(COURSE CODE 22MATC21/22MATS21/22MATE21)

CLASS NOTES FOR II SEM B.E.

MODULE 2

NUMERICAL METHODS-1

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SYLLABUS:

Solution of polynomial and transcendental equations: Regula-Falsi and Newton-Raphson methods (only formulae). Problems.

Interpolation :Finite differences, Interpolation using Newton's forward and backward difference formulae and Lagrange's interpolation formula (All formulae without proof). Problems.

Numerical integration: Trapezoidal,Simpson's $(1/3)^{rd}$ and $(3/8)^{th}$ rules(without proof). Problems.

Numerical solutions of Algebraic and Transcendental equations:

Definitions:

An equation involving algebraic terms is called an algebraic equation.

Example: The equation $x^3 - 2x - 5 = 0$ is an algebraic equation.

An equation involving non- algebraic terms such as logarithmic, exponential, and trigonometric is called an algebraic equation.

Example: The equations (i) $\sin x - \frac{1}{x} = 0$, (ii) $x^3 - 4e^{-x} + 5 = 0$ and (iii) $x \log_{10} x - 12 = 0$ are transcendental equations.

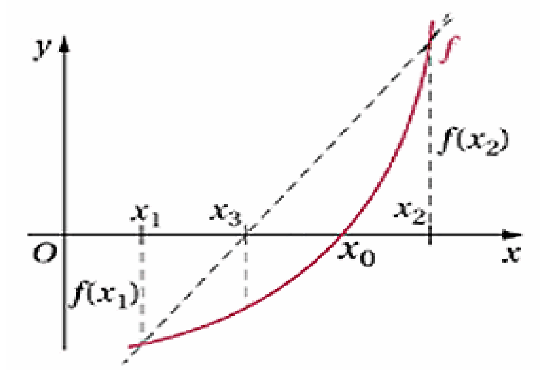
There is no direct method to solve the equations involving algebraic, logarithmic, exponential, and trigonometric functions. Such equations can be solved by an alternative method known as numerical method.

A numerical method of finding approximate roots of the equations is a repetitive type of process known as iteration process. In each step the result of the previous step is used, this process is repeated till the root is obtained to the desired accuracy.

Intermediate value Theorem:

If a function $f(x)$ is continuous in an interval $[a, b]$ such that $f(a)$ *and* $f(b)$ are of opposite signs, i.e., $f(a).f(b)< 0$, then there is a real root m of the equation $f(x) = 0$ lies between a and b.

Regula Falsi Method or Method of False Position:



This method is the graphical method of finding an approximate real root of an equation.
Let the graph of the equation f(x)=0 passes through two points $A[a, f(a)]$ and $B[b, f(b)]$ such that $f(a)$ *and* $f(b)$ are of opposite signs. Let m be a real

root of the equation $f(x) = 0$ lies between a and b, then m is the abscissa of the point where the graph of $y = f(x)$ meets the x-axis.
Join AB. If the chord AB meets the x-axis at a point whose abscissa is x_1, $then\ x_1$ is the first approximation of the root is given by the formula,

$$x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}.$$

If $f(a)\ and\ f(x_1)$ are of opposite signs, then the real root lies between a and x_1. Therefore the second approximation to the root is given by the formula,

$$x_2 = \frac{af(x_1)-x_1f(a)}{f(x_1)-f(a)}.$$

Otherwise, if $f(x_1)\ and\ f(b)$ are of opposite signs, then the real root lies between x_1 and b. Therefore the second approximation to the root is given by the formula,

$$x_2 = \frac{x_1f(b)-bf(x_1)}{f(b)-f(x_1)}.$$

This process is repeated until to get the solution to the desired accuracy.

Problems:

1. Find an approximate real root of the equation $x^3 - 2x - 5 = 0$ by Regula – Falsi method, correct to 3 decimal places.

Solution:
Given $f(x) = x^3 - 2x - 5$

$\therefore$ $f(0) = -5$, $f(1) = -6$. $\therefore$ $f(2) = -1 < 0$ and $f(3) = 16 > 0$.

$\therefore$ $f(2)\,f(3) < 0$. i.e., $f(2)$ and $f(3)$ are of opposite signs.

A real root lies in (2, 3). Now $f(2.1) = 0.061 > 0$.

$\therefore$ $f(2)$ and $f(2.1)$ are of opposite signs.

$\therefore$ Nearest root lies in (2, 2.1)

Step I: Take a=2 and b=2.1. $\therefore$ $f(a) = f(2) = -1$ and $f(b) = f(2.1) = 0.061$.

The first approximation is given by $x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$

$\therefore$ $x_1 = \frac{2(0.061)-(2.1)(-1)}{0.061-(-1)} = 2.094$

Step II: Now $f(2.094) = (2.094)^3 - 2(2.094) - 5 = -0.006 < 0$ and $f(2.1) = 0.061$.

$\therefore$ The root lies in (2.094, 2.1).

$\therefore$ Take x_1=2.094 and b=2.1. $\therefore$ $f(x_1) = f(2.094) = -0.006$ and

$f(b) = f(2.1) = 0.061.$

The second approximation is given by $x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)}$

$x_2 = \frac{(2.094)(0.061)-(2.1)(-0.006)}{0.061-(-0.006)} = 2.095.$

Step III: Now $f(2.095) = (2.095)^3 - 2(2.095) - 5 = 0.005 > 0$ and

$f(2.094) = -0.006.$ $\therefore$ The root lies in (2.094, 2.095).

$\therefore$ Take x_1=2.094 and x_2=2.095. $\therefore$ $f(x_1) = f(2.094) = -0.006$ and

$f(x_2) = f(2.095) = 0.005.$

The third approximation is given by $x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

$x_3 = \frac{(2.094)(0.005)-(2.095)(-0.006)}{0.005-(-0.006)} = 2.095.$

Thus the required approximate root correct to three decimal places is $x = 2.095$.

2. Find a real root of the equation $x \log_{10} x = 1.2$ by Regula – Falsi method, correct to four decimal places.

Solution:

Given $f(x) = x \log_{10} x - 1.2$

$\therefore$ $f(1) = -1.2,\ f(2) = -0.5979 < 0$ *and* $f(3) = 0.2314 > 0$

A real root lies in (2,3). Now f(2.7) = −0.0353 and f(2.8) = 0.052

$\therefore$ Nearest root lies in (2.7, 2.8)

First iteration: Take a = 2.7 and b = 2.8. $\therefore$ f(a) = −0.0353 and f(b) = 0.052.

The first approximation is given by $x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

$x_1 = \frac{(2.7)(0.052)-(2.8)(-0.0353)}{0.052+0.0353} = 2.7404.$

Second iteration : Now $f(2.7404) = -0.0002 < 0$ and f(b) = 0.052.

$\therefore$ The roots lies in (2.7404, 2.8).

Take $x_1 = 2.7404$ and b = 2.8. $\therefore$ f(x_1) = −0.0002 and f(b) = 0.052.

The second approximation is given by $x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)}$

$$x_2 = \frac{(2.7404)(0.052) - (2.8)(-0.0002)}{0.052 + 0.0002} = 2.7406$$

III iteration : Now $f(2.7406) = 0.0000$.

Thus the required approximate root is $x = 2.7406$.

3. By using the method of False -position, find a real root of the equation $cosx = 3x - 1$, correct to three decimal places.

Solution:

Given $f(x) = cosx - 3x + 1$. (x is in radians)

$\therefore \quad f(0) = 2 \ \ and \ \ f(1) = -1.46 < 0$

A real root lies in $(0, 1)$. Now $f(0.6) = 0.025$ and $f(0.7) = -0.335$

$\therefore$ Nearest root lies in $(0.6, 0.7)$

First iteration: Take $a = 0.6$ and $b = 0.7$.

$\therefore$ $f(a) = f(0.6) = 0.025$ and $f(b) = f(0.7) = -0.335$.

The first approximation is given by $x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$

$$x_1 = \frac{(0.6)(-0.335)-(0.7)(0.025)}{-0.335-0.025} = 0.607.$$

Second iteration : Now $f(0.607) = 0.000 < 0$.

Thus the required approximate root is $x = 0.607$.

4. Use Regula Falsi method to find the real root of $xe^x = cosx$ which lies in the interval (0, 1). Carry out four iterations.

Solution:

Given $f(x) = xe^x - cosx$.

$\therefore \ f(0) = -1 < 0 \ \ and \ \ \ f(1) = 2.178 > 0$.

$\therefore \ f(0)\, f(1) < 0$. i.e., $f(0)$ and $f(1)$ are of opposite signs.

A real root lies in $(0, 1)$. Now $f(0.4) = -0.3243 < 0$ and $f(0.6) = 0.268$.

$\therefore$ $f(0.4)$ and $f(0.6)$ are of opposite signs. $\therefore$ Nearest root lies in $(0.4, 0.6)$.

Step I: Take a=0.4 and b=0.6. $\therefore \ f(a) = f(0.4) = -0.3243$ and

$f(b) = f(0.6) = 0.268$.

The first approximation is given by $x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$

$\therefore \quad x_1 = \frac{0.4(0.268)-(0.6)(-0.3243)}{0.268-(-0.3243)} = 0.5095.$

Step II: Now $f(0.5095) = -0.0249 < 0$ and $f(0.6) = 0.268 > 0.$

$\therefore$ The root lies in (0.5095, 0.6)

$\therefore$ Take x_1=0.5095 and b=0.6. $\therefore$ $f(x_1) = f(0.5095) = -0.0249$ and

$f(b) = f(0.6) = 0.268.$

The second approximation is given by $x_2 = \frac{x_1 f(b)-bf(x_1)}{f(b)-f(x_1)}$

$x_2 = \frac{(0.5095)(0.268)-(0.6)(-0.0249)}{0.268-(-0.0249)} = 0.5172.$

Step III: Now $f(0.5172) = -0.0017 < 0$ and $f(0.6) = 0.268 > 0.$

$\therefore$ The root lies in (0.5172, 0.6).

$\therefore$ Take x_2=*0.5172* and b=0.6.

$\therefore$ $f(x_2) = f(0.5172) = -0.0017$ and $f(b) = f(0.6) = 0.268.$

The third approximation is given by $x_3 = \frac{x_2 f(b)-bf(x_2)}{f(b)-f(x_2)}$

$x_3 = \frac{(0.5172)(0.268)-(0.6)(-0.0017)}{0.268-(-0.0017)} = 0.5177.$

Step IV: Now $f(0.5177) = -0.0001 < 0$ and $f(0.6) = 0.268 > 0.$

$\therefore$ The root lies in (0.5177, 0.6).

$\therefore$ Take x_3=*0.5177* and b=0.6.

$\therefore$ $f(x_3) = f(0.5177) = -0.0001$ and $f(b) = f(0.6) = 0.268.$

The third approximation is given by $x_4 = \frac{x_3 f(b)-bf(x_3)}{f(b)-f(x_3)}$

$x_4 = \frac{(0.5177)(0.268)-(0.6)(-0.0001)}{0.268-(-0.0001)} = 0.5177.$

Thus the required approximate root correct to three decimal places is $x = 0.5177$.

5. Find the fourth-root of 12 correct to four decimal places using the method of False position.

Solution:

Let $x = \sqrt[4]{12}$ $\therefore$ $x^4 = 12$ or $x^4 - 12 = 0$. $\therefore$ Take $f(x) = x^4 - 12$.

$f(0) = -12 < 0,\ f(1) = -11 < 0,\ f(2) = 4 > 0$

$\therefore$ A real root lies in (1, 2). Further $f(1.8) = -1.5024$ and $f(1.9) = 1.0321$.

$\therefore$ The root lies in (1.8, 1.9).

Step I: Take a=1.8 and b=1.9. $\therefore$ $f(a) = f(1.8) = -1.5024$ and

$f(b) = f(1.9) = 1.0321.$

The first approximation is given by $x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$

$x_1 = \frac{1.8(1.0321)-(1.9)(-1.5024)}{1.0321+1.5024}$ = 1.8593

Step II: Now $f(1.8593) = -0.0492 < 0$ and $f(1.9) = 1.0321$.

$\therefore$ The root lies in (1.8593, 1.9).

Take x_1=1.8593 and b=1.9.

$\therefore$ $f(x_1) = f(1.8593) = -0.0492$ and $f(b) = f(1.9) = 1.0321$.

The second approximation is given by $x_2 = \frac{x_1 f(b)-bf(x_1)}{f(b)-f(x_1)}$

$\therefore$ $x_2 = \frac{1.8593(1.0321)-(1.9)(-0.0492)}{1.0321+0.0492}$ = 1.8612.

Step III: Now $f(1.8612) = -0.0003 < 0$ and $f(1.9) = 1.0321$.

$\therefore$ The root lies in (1.8612 , 1.9)

Take x_2=1.8612 and b=1.9.

$\therefore$ $f(x_2) = f(1.8612) = -0.0003$ and $f(b) = f(1.9) = 1.0321$.

The third approximation is given by $x_3 = \frac{x_2 f(b)-bf(x_2)}{f(b)-f(x_1)}$

$\therefore$ $x_3 = \frac{1.8612(1.0321)-(1.9)(-0.0003)}{1.0321+0.0003}$ = 1.8612.

Thus the required approximate root is $x = 1.8612$. Hence $\sqrt[4]{12}$= 1.8612.

HOMEWORK:

1. Find the real root of the equation $x^4 - x - 9 = 0$ by regula falsi method correct to three places of decimal in the interval (1, 2).

2. Obtain the real root of the equation $xe^x = 2$ correct to four decimal places by using the Regula- Falsi iterative method.

3. Find a real root of the equation $2x - log_e x = 12$ by using Regula Falsi method. Carry out four iterations

4. Find a real root of the equation tan x + tanhx =0 lies between 2 and 3 by Regula – Falsi method correct to 4 decimal places.

5. Using Regula-Falsi iterative method find a negative root of the equation $xe^x = sinx$ correct to four decimal places (perform three iterations).

6. Find the fourth-root of 32 correct to three decimal places using the method of false position.

Newton-Raphson Method (N-R method):

The Newton-Raphson method, or Newton's Method is an iterative method of finding an approximate root of an equation f(x)= 0.
Let x_0 be the initial aproximation to the root of the equation f(x)= 0. Then the first approximation x_1 to the root of the equation f(x)= 0 is given by the formula,

$x_1 = x_0 - \frac{f(x_0)}{f^{'}(x_0)}$. Simillerly, the second approximation x_2 to the root of the equation

f(x)= 0 is given by the formula,

$x_2 = x_1 - \frac{f(x_1)}{f^{'}(x_1)}$ and so on.

In general the n th approximation x_n to the root of the equation

f(x)= 0 is given by the formula,

$x_{n+1} = x_n - \frac{f(x_n)}{f^{'}(x_n)}$ for $n = 0,1,2, \dots \dots \dots \dots$

This is known as **Newton-Raphson Formula.** The process is continued until to get the root to the desired accuracy.

Problems:

1. Find the real root of the equation $x^4 - x - 9 = 0$ by Newton-Raphson method correct to three places of decimal.

Solution:

Let $f(x) = x^4 - x - 9$ and $f^{'}(x) = 4x^3 - 1.$

$\therefore\ f(0) = -9,\ \ f(1) = -9 < 0\ \ and\ \ f(2) = 6\ > 0.$

$\therefore\ f(1)\, f(2) < 0.$ i.e., $f(1)$ and $f(2)$ are of opposite signs.

A real root lies in $(1,2)$ and it will be in the neighbourhood of 2. Therefore take $x_0 = 2.$

The first approximation is given by

$$x_1 = x_0 - \frac{f(x_0)}{f^{\,|}(x_0)} = 2 - \frac{2^4-2-9}{4(2)^3-1} = 2 - \frac{5}{31} = 1.839.$$

$$x_2 = x_1 - \frac{f(x_1)}{f^{\,|}(x_1)} = 1.839 - \frac{(1.839)^4-1.839-9}{4(1.839)^3-1} = 1.814.$$

$$x_3 = x_2 - \frac{f(x_2)}{f^{\,|}(x_2)} = 1.814 - \frac{(1.814)^4-1.814-9}{4(1.814)^3-1} = 1.813$$

$$x_4 = x_3 - \frac{f(x_3)}{f^{\,|}(x_3)} = 1.813 - \frac{(1.813)^4-1.813-9}{4(1.813)^3-1} = 1.813$$

$\therefore$ The real root is 1.813.

2. Using Newton-Raphson method find the real root of the equation $3x = cosx + 1$.

Solution:

Let $f(x) = 3x - cosx - 1$ and $f^{\,|}(x) = 3 + \sin x.$

$\therefore\ f(0) = -2\ < 0\ \ and\ \ f(1) = 1.46 > 0.$

$\therefore\ f(0)\, f(1) < 0.$ i.e., $f(0)$ and $f(1)$ are of opposite signs.

A real root lies in $(0, 1)$ and it will be in the neighbourhood of 1. Therefore take $x_0 = 0.6.$

The first approximation is given by

$$x_1 = x_0 - \frac{f(x_0)}{f^{\,|}(x_0)} = 0.6 - \frac{3(0.6)-cos(0.6)-1}{3+\sin(0.6)} = 0.6071.$$

$$x_2 = x_1 - \frac{f(x_1)}{f^{\,|}(x_1)} = 0.6071 - \frac{3(0.6071)-cos0.6071-1}{3+\sin 0.6071} = 0.6071.$$

$\therefore$ The real root is 0.6071.

3. Find a real root of the equation $x^3 + x^2 + 3x + 4 = 0$ applying N-R method.

Carryout two iterations.

Solution:

Let $f(x) = x^3 + x^2 + 3x + 4$ and $f^{\,|}(x) = 3x^2 + 2x + 3$

$f(0) = 4,\ f(-1) = 1,\ f(-2) = -6 < 0$

A real root lies in $(-2, -1)$ and let $x_0 = -1$

The first approximation is given by

$$x_1 = x_0 - \frac{f(x_0)}{f^{\mathrm{I}}(x_0)} = -1 - \frac{[(-1)^3+(-1)^2+3(-1)+4]}{3-2+3} = -1 - \frac{1}{4} = -1.25$$

$$x_2 = x_1 - \frac{f(x_1)}{f^{\mathrm{I}}(x_1)} = -1.25 - \frac{[(-1.25)^3+(-1.25)^2+3(-1.25)+4]}{[3(-1.25)^2+2(-1.25)+3]} = -1.2229$$

∴ The real root is -1.2229.

4. Using Newton – Raphson method find the real root of $xlog_{10}x = 1.2$ correct to four decimal places which is near $x = 2$

Solution:

Let $f(x) = xlog_{10}x - 1.2$ and $f^{\mathrm{I}}(x) = x\left(\frac{1}{x\,log10}\right) + log_{10}x$

∴ $f^{\mathrm{I}}(x) = log_e 10 + log_{10}x = 0.4343 + log_{10}x$. Take $x_0 = 2$.

The first approximation is given by

$$x_1 = x_0 - \frac{f(x_0)}{f^{\mathrm{I}}(x_0)} = 2 - \frac{2log_{10}2 - 1.2}{0.4343 + log_{10}2} = 2.81.$$

$$x_2 = x_1 - \frac{f(x_1)}{f^{\mathrm{I}}(x_1)} = 2.81 - \frac{(2.81)log_{10}(2.81) - 1.2}{0.4343 + log_{10}(2.81)} = 2.741.$$

$$x_3 = x_2 - \frac{f(x_2)}{f^{\mathrm{I}}(x_2)} = 2.741 - \frac{(2.741)log_{10}(2.741) - 1.2}{0.4343 + log_{10}(2.741)} = 2.7406.$$

$$x_4 = x_3 - \frac{f(x_3)}{f^{\mathrm{I}}(x_3)} = 2.7406 - \frac{(2.7406)log_{10}(2.7406) - 1.2}{0.4343 + log_{10}(2.7406)} = 2.7406.$$

∴ The real root is 2.7406.

5. Using Newton – Raphson method find the real root of $xsinx + cosx = 0$ correct to four decimal places which is near $x = \pi$.

Solution:

Let $f(x) = xsinx + cosx$ and $f^{\mathrm{I}}(x) = xcosx + sinx - sinx = xcosx$. Take $x_0 = \pi$.

The first approximation is given by

$$x_1 = x_0 - \frac{f(x_0)}{f^{\mathrm{I}}(x_0)} = \pi - \frac{\pi sin\pi + cos\pi}{\pi cos\pi} = 2.8233.$$

$$x_2 = x_1 - \frac{f(x_1)}{f^{\mathrm{I}}(x_1)} = 2.8233 - \frac{(2.8233)sin(2.8233) + cos(2.8233)}{(2.8233)cos(2.8233)} = 2.7986.$$

$x_3 = x_2 - \frac{f(x_2)}{f^{\prime}(x_2)} = 2.7986 - \frac{(2.7986)sin(2.7986) + cos(2.7986)}{(2.7986)cos(2.7986)} = 2.7984.$

$x_4 = x_3 - \frac{f(x_3)}{f^{\prime}(x_3)} = 2.7986 - \frac{(2.7986)sin(2.7986) + cos(2.7986)}{(2.7986)cos(2.7986)} = 2.7984.$

$\therefore$ The real root is 2.7984.

6. Evaluate $\sqrt[3]{24}$ by using Newton's iteration method correct to four decimal places.

Solution:

Let $x = \sqrt[3]{24}$. $\therefore$ $x^3 = 24$. $\therefore$ $x^3 - 24 = 0$.

$\therefore$ $f(x) = x^3 - 24$ and $f^{\prime}(x) = 3x^2$.

$f(0) = -24,\ f(1) = 23,\ f(2) = -16 < 0$ and $f(3) = 3 > 0$

A real root lies in $(2, 3)$. Taske $x_0 = 3$.

The first approximation is given by

$x_1 = x_0 - \frac{f(x_0)}{f^{\prime}(x_0)} = 3 - \frac{3^3 - 24}{3(3^2)} = 2.8889.$

$x_2 = x_1 - \frac{f(x_1)}{f^{\prime}(x_1)} = 2.8889 - \frac{(2.8889)^3 - 24}{3(0.8889)^2} = 2.8845.$

$x_3 = x_2 - \frac{f(x_2)}{f^{\prime}(x_2)} = 2.8845 - \frac{(2.8845)^3 - 24}{3(0.8845)^2} = 2.8845.$

$\therefore$ The real root is 2.8845. $\therefore$ $\sqrt[3]{24} = 2.8845$.

Home work:

1. **Using N-R method, find a real root of the equation $e^x = x^2 + \cos 25x$ near $x = 4.5$.**
2. **By Newton–Raphson method, find the real root of the equation $x^3 - 3x - 5 = 0$ correct to three decimal places.**
3. **Explain intermediate value property geometrically and find a real root of the equation $x + log_{10}x = 3.375$ correct to four places of decimals using Newton–Raphson iterative method (perform three iterations).**
4. **Using the Newton-Raphson method, find an approximate root of the equation $e^x - 3x = 0$.**

5. **Find a real root of the equation $xe^x - 2 = 0$ near $x = 1$ by using Newton-Raphson method.**

6. **By using Newton–Raphson method, find the real root of the equation $xe^x = cosx$ correct to four decimal places.**

7. **Using Newton – Raphson method find the real root of $x\,tanx + 1 = 0$ correct to four decimal places which is near $x = \pi$.**

8. **Use Newton-Raphson method to find the value of $\sqrt[3]{37}$ correct to 3 decimal places.**

Finite Differences and Interpolation:

Interpolation:

Let $y_0, y_1, y_2, \dots\dots y_n$ be a set of values of a function $y = f(x)$ corresponding to the values $x_0, x_1, x_2, \dots\dots\dots x_n$ of x respectively. Then the process of finding the value of y for any given value of x between x_0 and x_n is called **interpolation**. The process of finding the value of y outside the interval $[x_0, x_n]$ is called **extrapolation.**

In general the concept of interpolation includes extrapolation also.

Suppose the values of $y = f(x)$ is tabulated for the equally spaced values $x = x_0,\ x_1 = x_0 + h,\ x_2 = x_1 + h, \dots\dots x_n = x_{n-1} + h$ giving $y = y_0, y_1, y_2, \dots\dots y_n$. To find the values of $f(x)$ or $f'(x)$ for some intermediate values of x, the following types of differences are useful.

(i) Forward Differences:

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \dots\dots, y_n - y_{n-1}$, denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \dots, \Delta y_{n-1}$ respectively are called the first forward differences, where Δ is called the forward difference operator. In general, the first forward differences are given by

$\Delta y_r = \Delta y_{r-1} - \Delta y_r$. Where r = 0, 1, 2, ..., n−1. Similarly, the second forward differences are defined by $\Delta^2 y_r = \Delta^2 y_{r-1} - \Delta^2 y_r$. The p[th] forward differences are defined by $\Delta^p y_r = \Delta^p y_{r-1} - \Delta^p y_r$.

(ii) Backward Differences:

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \dots\dots, y_n - y_{n-1}$, denoted by $\nabla y_1, \nabla y_2, \nabla y_3, \dots, \nabla y_n$ respectively are called the first backward differences, where ∇ is called the backward difference operator. In general, the first backward differences are given by

$\nabla y_r = \nabla y_r - \nabla y_{r-1}$. Where r = 0, 1, 2, ……., n−1. Similarly, the second backward differences are defined by $\nabla^2 y_r = \nabla^2 y_{r-1} - \nabla^2 y_r$. The p[th] backward differences are defined by $\nabla^p y_r = \nabla^p y_r - \nabla^p y_{r-1}$.

The following table is called **Forward Differences table.**

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	…………..	$\Delta^n y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$			
.	.	.		.	.	
.	.	.	.	.	.	$\Delta^n y_0$
.	.	.	.	.		
.	.	.	.			
.	.	.	.	$\Delta^3 y_{n-3}$		
x_{n-1}	y_{n-1}		$\Delta^2 y_{n-2}$			
		Δy_{n-1}				
x_n	y_n					

The first entries in the table namely Δy_0 , $\Delta^2 y_0$, $\Delta^3 y_0$ ……….. $\Delta^n y_0$ are called the leading forward differences .

The following table is called **Backward Differences table.**

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	…………..	$\nabla^n y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$			
.	.	.		.	.	
.	.	.	.	.	.	$\nabla^n y_n$
.	.	.	.	.		
.	.	.	.			
.	.	.	.	$\nabla^3 y_n$		
x_{n-1}	y_{n-1}		$\nabla^2 y_n$			
		∇y_n				
x_n	y_n					

The last entries in the table namely ∇y_n , $\nabla^2 y_n$, $\nabla^3 y_n$ ……….. $\nabla^n y_n$ are called the leading backward differences .

Newton's Interpolation formulae for equal intervals:

1. Newton's forward interpolation formula(NFIF):

Let $y_0, y_1, y_2, \dots \dots . y_n$ be a set of values of a function $y = f(x)$ corresponding to the values $x_0,\ x_1 = x_0 + h,\ x_2 = x_1 + h, \dots \dots . . x_n = x_{n-1} + h$ of x respectively. Then the value of $y = f(x)$ at $x = x_0 + ph$, that is $y_p = f(x_0 + ph)$ is approximately given by

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\ldots\ldots$$

Where p is any real number.

2. Newton's backward interpolation formula(NBIF):

Let $y_0, y_1, y_2, \ldots \ldots . y_n$ be a set of values of a function $y = f(x)$ corresponding to the values $x_0,\ x_1 = x_0 + h,\ x_2 = x_1 + h, \ldots \ldots . . x_n = x_{n-1} + h$ of x respectively. Then the value of $y = f(x)$ at $x = x_n + ph$, that is $y_p = f(x_n + ph)$ is approximately given by

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots\ldots\ldots\ldots\ldots\ldots$$

Where p is any real number.

Problems:

1. Find the cubic polynomial which takes the following values:

x	0	1	2	3
$f(x)$	1	2	1	10

Hence evaluate $f(4)$.

Solution:

Difference Table:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	**1**			
		1		
1	2		**−2**	
		−1		**12**
2	1		10	
		9		
3	10			

Take $x_0 = 0,\ and\ \ h = 1\ \ \therefore p = \frac{x - x_0}{h} = \frac{x-0}{1} = x.$

$y_0 = 1,\ \Delta y_0 = 1,\ \Delta^2 y_0 = -2,\ \Delta^3 y_0 = 12.$

Using NFIF,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots$$

$$\therefore\ y = f(x) = 1 + x(1) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12)$$

$\therefore\ f(x) = 2x^3 - 7x^2 + 6x + 1. \qquad \therefore\ \ f(4) = 41.$

2. The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

x =height	100	150	200	250	300	350	400
y =distance	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when $x = 218ft$ and $410ft$.

Solution:

Difference table:

x	$y = f(x)$	I diff	II diff	III diff	IV diff	V diff	VI diff
100	**10.63**						
		2.40					
150	13.03		**−0.39**				
		2.01		**0.15**			
200	15.04		−0.24		**−0.07**		
		1.77		0.08		**0.02**	
250	16.81		−0.16		−0.05		**0.02**
		1.61		0.03		**0.04**	
300	18.42		−0.13		**−0.01**		
		1.48		**0.02**			
350	19.90		**−0.11**				
		1.37					
400	**21.27**						

(i) To find $f(218)$, take $x_0 = 100$, $x = 218$ and $h = 50$ $\therefore p = \frac{x-x_0}{h} = \frac{218-100}{50} = 2.36.$

$y_0 = 10.63$, $\Delta y_0 = 2.40$, $\Delta^2 y_0 = -0.39$, $\Delta^3 y_0 = 0.15$, $\Delta^4 y_0 = -0.07$, $\Delta^5 y_0 = 0.02$, $\Delta^6 y_0 = 0.02$.

Using NFIF,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots\dots\dots$$

$$\therefore f(218) = 10.63 + (2.36)(2.40) + \frac{(2.36)(2.36-1)}{2!}(-0.39) + \frac{(2.36)(2.36-1)(2.36-2)}{3!}(0.15)$$

$$+ \frac{(2.36)(2.36-1)(2.36-2)(2.36-3)}{4!}(-0.07) + \frac{(2.36)(2.36-1)(2.36-2)(2.36-3)(2.36-4)}{5!}(0.02)$$

$$+ \frac{(2.36)(2.36-1)(2.36-2)(2.36-3)(2.36-4)(2.36-5)}{6!}(0.02).$$

$\therefore f(218) = 15.696.$

(ii) To find $f(410)$, take $x_n = 400, x = 410$ and $h = 50$ $\therefore p = \frac{x-x_n}{h} = \frac{410-400}{50} = 0.2.$

$y_n = 21.27$, $\nabla y_n = 1.37$, $\nabla^2 y_n = -0.11$, $\nabla^3 y_n = 0.02$, $\nabla^4 y_n = -0.01$, $\nabla^5 y_n = 0.04$,

$\nabla^6 y_n = 0.02.$

By using NBFI,

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore\ f(410) = 21.27 + (0.2)(1.37) + \frac{(0.2)(0.2+1)}{2!}(-0.11) + \frac{(0.2)(0.2+1)(0.2+2)}{3!}(0.02)$$

$$+\frac{(0.2)(0.2+1)(0.2+2)(0.2+3)}{4!}(-0.01) + \frac{(0.2)(0.2+1)(0.2+2)(0.2+3)(0.2+4)}{5!}(0.04)$$

$$+\frac{(2.36)(2.36+1)(2.36+2)(2.36+3)(2.36+4)(2.36+5)}{6!}(0.02).$$

$\therefore\ f(410) = 21.53.$

3. From the following table ,estimate the number of students who obtained marks

(i) less than 45 marks (ii)between 40 and 45:

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

Solution:

Rewrite the given table with $f(x)$ repersenting the number of students less than x marks as follows:

Marks less than (x)	40	50	60	70	80
No. of Students	31	73	124	159	190

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	**31**				
		42			
50	73		**9**		
		51		**−25**	
60	124		−16		**37**
		35		12	
70	159		−4		
		31			
80	190				

(i) Take $x_0 = 40,\ x = 45\ and\ h = 10\ \ \therefore p = \frac{x - x_0}{h} = \frac{45-40}{10} = 0.5.$

$y_0 = 31,\ \Delta y_0 = 42,\ \Delta^2 y_0 = 9,\ \Delta^3 y_0 = -25,\ \Delta^4 y_0 = 37.$

Using NFIF,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore\ f(45) = 31 + (0.5)42 + \frac{(0.5)(0.5-1)}{2!}(9) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(-25)$$

$$+\frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}(37).$$

$f(45) = 47.86 \approx 48.$

$\therefore$ The number of students obtaining less than 45 marks is 48.

(ii) The number of students obtaining the marks between 40 and 45 = $f(45) - f(40)$

$= 48 - 31 = 17$ students

4. The area A of a circle of diameter d is given for the following values:

d	80	85	90	95	100
A	5026	5674	6362	7088	7854

Find the area of the circle of the diameter 105 by using suitable interpolation formula.

Solution:

Let $d = x$ *and* $A = y$. Since 105 is nearer to 100, we use Newton's backward interpolation formula. Therefore take $x_n = 100,\ x = 105\ \ and\ \ h = 5.\ \ \therefore p = \frac{x-x_n}{h} = \frac{105-100}{5} = 1.$

Difference Table:

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		648			
85	5674		40		
		688		-2	
90	6363		38		**4**
		726		**2**	
95	7088		**40**		
		766			
100	**7854**				

$\therefore\ y_n$=7854, ∇y_n= 766, $\nabla^2 y_n$= 40, $\nabla^3 y_n$= 2, $\nabla^4 y_n = 4.$

By using NBFI,

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots\dots\dots\dots$$

$$\therefore\ f(105) = 7854 + (1)(766) + \frac{(1)(1+1)}{2!}(40) + \frac{(1)(1+1)(1+2)}{3!}(2) + \frac{(1)(1+1)(1+2)(1+3)}{4!}(4).$$

$\therefore\ f(105) = 8666.$

$\therefore$ The are A = 8666 when the diameter d = 105.

HOME WORK:

1. Find y(1.4) from the given data using appropriate interpolation formula

x	1	2	3	4	5
y	10	26	58	112	194

2. Using Newton's Forward interpolation formula, find the interpolating polynomial for the following data:

x	4	6	8	10
y	1	3	8	16

Hence evaluate y for x = 5.

3. Find y(3.8) from the given data using Gregory Newton backward interpolation formula

x	0	1	2	3	4
y	1	1.5	2.2	3.1	4.6

4. In the table below, the values of y are consecutive terms of a series of which 23.6 is the 6^{th} term. Find the first and tenth terms of the series:

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

5. Find $f(2.5)$ by using Newton's backward interpolation formula given that

$f(0) = 7.4072$, $f(1) = 7.5854$, $f(2) = 7.6922$, $f(3) = 7.8119$,

$f(4) = 7.9252$.

6. Given $\sin 45^0 = 0.7071$, $\sin 50^0 = 0.7660$, $\sin 55^0 = 0.8192$,

$\sin 60^0 = 0.8660$, find $\sin 57^0$ using Newton backward interpoliation formula.

7. A survey conducted in a locality reveals the following information:

Income per day(Rs)	0-100	100-200	200-300	300-400
Number of persons	9	30	35	42

Estimate the number of persons getting income between Rs.100 and Rs.150.

8. The population of a town is given by the table

year	1951	1961	1971	1981	1991
Population in thousands	19.96	39.65	58.81	77.21	94.61

Using Newton's Backward interpolation formula, calculate the increase of the population from the year 1955 to 1985.

Lagrange's Interpolation formula(unequal intervals):

Let $y_0, y_1, y_2, \ldots \ldots, y_n$ be a set of values of a function $y = f(x)$ corresponding to the values $x_0, x_1, x_2, \ldots \ldots \ldots \ldots, x_n$ of x which are not necessarily equally spaced. Then the value of $y = f(x)$ at any value of x given by

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots\ldots(x_0-x_n)}\, y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots\ldots(x_1-x_n)}\, y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\ldots\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots\ldots(x_2-x_n)}\, y_2 + \ldots\ldots + \frac{(x-x_0)(x-x_1)(x-x_2)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots\ldots(x_n-x_{n-1})}\, y_n.$$

Problems:

1. Given the values

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Evaluate $f(9)$ using Lagrange's Interpolation formula.

Solution:

Given $x_0 = 5,\ x_1 = 7,\ x_2 = 11,\ x_3 = 13,\ x_4 = 17.$

$y_0 = 150,\ y_1 = 392,\ y_2 = 1452,\ y_3 = 2366,\ y_4 = 5202.$ Take x = 9.

We have Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}\, y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}\, y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}\, y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}\, y_3$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_4)}\, y_4$$

$$\therefore\quad f(9) = \frac{(2)(-2)(-4)(-8)\times 150}{(-2)(-6)(-8)(-12)} + \frac{(4)(-2)(-4)(-8)\times 392}{(2)(-4)(-6)(-10)} + \frac{(4)(2)(-4)(-8)\times 1452}{(6)(4)(-2)(-6)}$$

$$+ \frac{(4)(2)(-2)(-8)\times 2366}{(8)(6)(2)(-4)} + \frac{(4)(2)(-2)(-4)\times 5202}{(12)(10)(6)(4)}$$

$$\therefore\quad f(9) = -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} \qquad \therefore\quad f(9) = 810.$$

2. Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for

x	0	1	2	5
$f(x)$	2	3	12	147

Solution:

Given $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 5.\ y_0 = 2,\ y_1 = 3,\ y_2 = 12,\ y_3 = 147.$

We have Lagrange's interpolation formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)\,(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\ y_0 + \frac{(x-x_0)(x-x_2)\,(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\ y_1 + \frac{(x-x_0)(x-x_1)\,(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\ y_2$$

$$+ \frac{(x-x_0)(x-x_1)\,(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\ y_3$$

$$\therefore\quad f(x) = \frac{(x-1)(x-2)\,(x-5)}{(-1)(-2)(-5)}\ 2 + \frac{(x-0)(x-2)\,(x-5)}{(1)(-1)(-4)}\ 3 + \frac{(x-0)(x-1)\,(x-5)}{(2)(1)(-3)}\ 12 + \frac{(x-0)(x-1)\,(x-2)}{(5)(4)(3)}\ 147.$$

[Use the formula (x −a)(x −b)(x−c) =$x^3 - x^2$(a + b + c) +x(ab + bc + ca) – abc]

$$\therefore\quad f(x) = \frac{1}{20}\,(20x^3 + 20x^2 - 20x + 40)$$

$$\therefore\quad f(x) = x^3 + x^2 - x + 2$$

$$\therefore\quad f(3) = 3^3 + 3^2 - 3 + 2 = 35.$$

3. Applying Lagrange's formula find y *when* $x = 6$ given the data

x	2	4.4	7.9
y	20	30	40

Solution:

Given $x_0 = 2,\ x_1 = 4.4,\ x_2 = 7.9.\ y_0 = 20,\ y_1 = 30,\ y_2 = 40$. Take x = 6.

We have Lagrange's inverse interpolation formula,

$$y = \frac{(x-x_1)(x-x_2)\,(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\ y_0 + \frac{(x-x_0)(x-x_2)\,(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\ y_1 + \frac{(x-x_0)(x-x_1)\,(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\ y_2$$

$$\therefore\quad y = \frac{(1.6)(-1.9)20}{(-2.4)(-5.9)} + \frac{(4)(-1.9)30}{(2.4)(-3.5)} + \frac{(4)(1.6)40}{(5.9)(3.5)}$$

$$\therefore\quad y(6) = 35.2462.$$

4. A curve passes through the point (0, 18), (1, 10), (3, −18) and (6, 90). Find the slope of the curve at x = 2.

Solution:

Given $x_0 = 0,\ x_1 = 1,\ x_2 = 3,\ x_3 = 6.\ y_0 = 18,\ y_1 = 10,\ y_2 = -18,\ y_3 = 90.$

We have Lagrange's interpolation formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)\,(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\ y_0 + \frac{(x-x_0)(x-x_2)\,(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\ y_1 + \frac{(x-x_0)(x-x_1)\,(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\ y_2$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\,y_3$$

$$\therefore y = \frac{(x-1)(x-3)(x-6)}{(-1)(-3)(-6)}\,18 + \frac{(x-0)(x-3)(x-6)}{(1)(-2)(-5)}\,10 + \frac{(x-0)(x-1)(x-6)}{(3)(2)(-3)}\,(-18) + \frac{(x-0)(x-1)(x-3)}{(6)(5)(3)}\,90.$$

[Use the formula (x −a)(x −b)(x−c) =$x^3 - x^2$(a + b + c) +x(ab + bc + ca) – abc]

$\therefore\quad y = (-x^3 + 10x^2 - 27x + 18) + (x^3 - 9x^2 + 18x) + (x^3 - 7x^2 + 6x)$

$+ (x^3 - 4x^2 + 3x).$

$\therefore\quad y = 2x^3 - 10x^2 + 18.$ Differentiate w. r. t. x.

$\therefore\quad \frac{dy}{dx} = 6x^2 - 20x.$

$\therefore$ The slope of the curve at x = 2 is $\left(\frac{dy}{dx}\right)_{x=2} = 6(2^2) - 20(2) = -16.$

HOMEWORK:

1. If y(1) = –3, y(3) = 9, y(4) = 30, y(6) = 132 find Lagrange's interpolation polynomial that takes on these values. Hence find y(5).

2. Use Lagrange's interpolation formula to find $f(4)$ given

x	0	2	3	6
y	–4	2	14	158

3. Use Lagrange's interpolation formula to find $f(4)$ given

x	5	6	9	11
y	12	13	14	16

4. The following table gives the normal weights of babies during first eight months of life

Age (in months)	0	2	5	8
Weight (in pounds)	6	10	12	16

Estimate the weight of the baby at the age of seven months using Lagrange's interpolation formula.

5. Using the Lagrange's formula, find $f(5)$ from the following table:

x	0	1	2	3	4
$f(x)$	3	6	11	18	27

Numerical Integration:

The process of obtaining approximate value of the definite integral I = $\int_a^b f(x)\,dx$ from the set of tabulated values of the integrand f(x) is called the numerical integration.

Divide the interval [a, b] into n sub intervals of width h so that x_0= a, $x_1 = x_0$ + h, x_2= x_1+ h, x_3= x_2 + h, ……..., $x_n = x_{n-1} + h$ = b. Let y_0, y_1, y_2, y_3……., y_{n-1}, y_n be the n + 1 values of y of the function y = f(x) corresponding to the values x_0, x_1, x_2, x_3……, x_{n-1}, x_n of x, then the integral I = $\int_a^b f(x)\,dx$ is obtained by using the suitable formula as follows.

Trapezoidal rule:

Formula:

$$I = \int_a^b f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$$

Where h = $\frac{b-a}{n}$. This is known as **Trapezoidal rule.**

Problems:

1. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using trapezoidal rule.

Soln:

x	0	1	2	3	4	5	6
y	**1**	**0.5**	**0.2**	**0.1**	**0.0588**	**0.0385**	**0.027**
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Here a = 0 and b = 6. Take n = 6. ∴ h = $\frac{6-0}{6}$ = 1.

$$I = \int_a^b f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$$

$$I = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}[(1 + \mathbf{0.027}) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)]$$

= 1.1408

2. A curve is drawn to pass through the points given by the following table

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	2	2.4	2.7	2.8	3	2.6	2.1

Using Trapezoidal rule, estimate the area bounded by the curve.

Soln:

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	2	2.4	2.7	2.8	3	2.6	2.1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Here a = 1 and b = 4. Take n = 6. ∴ h = $\frac{4-1}{6}$ = 0.5

$$I = \int_a^b y\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$$

$$I = \int_1^4 y\,dx = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$I = \frac{0.5}{2}[(2+2.1)+2(2.4+2.7+2.8+3+2.6)]$
$= 7.775$

3. Given that

x:	4.0	4.2	4.4	4.6	4.8	5.0	5.2
Log x:	**1.3863**	**1.4351**	**1.4816**	**1.5261**	**1.5686**	**1.6094**	**1.6484**

Evaluate $\int_4^{5.2} logx\ dx$

Soln:

x:	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$y = logx$	**1.3863**	**1.4351**	**1.4816**	**1.5261**	**1.5686**	**1.6094**	**1.6484**
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Here a = 4 and b = 5.2, Take n = 6. ∴ h = $\frac{5.2-4}{6}$ = 0.2.

$I = \int_a^b f(x)\,dx = \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})]$

$I = \int_4^{5.2} y\,dx\ \frac{h}{2}[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]$

$I = \frac{0.2}{2}[(1.3863+1.6484)+2(1.4351+1.4816+1.5261+1.5686+1.6094)]$
$= 1.8276$

HOMEWORK:

1. **Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using trapezoidal rule taking h = ¼.**

Ans: 0.7854

Simpson's one third (1/3rd) rule:

If n is a multiple of 2 i.e., n is even, then

$I = \int_a^b f(x)\,dx = \frac{h}{3}[(y_0+y_n)+4(y_1+y_3+\cdots+y_{n-1})+2(y_2+y_4+\cdots+y_{n-2})].$

Where $h = \frac{b-a}{n}$. This is known as **Simpson's one third (1/3rd) rule.**

Problems:

1. Use Simpson's one third rule to find $\int_0^{0.6} e^{-x^2}$ dx by taking 6 sub intervals.

Solution:

Let f(x) = e^{-x^2}. Here a = 0 and b = 0.6. Take n = 6. ∴ h = $\frac{0.6-0}{6}$ = 0.1.

x	x_0= 0	x_1= 0.1	x_2= 0.2	x_3= 0.3	x_4= 0.4	x_5= 0.5	x_6= 0.6
$y = e^{-x^2}$	y_0= 1	y_1= 0.99	y_2= 0.9608	y_3= 0.9139	y_4= 0.8521	y_5= 0.7788	y_6= 0.6977

Using Simpson's one third rule, we have, I = $\int_0^{0.6} e^{-x^2} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

∴ I = $\frac{0.1}{3}[(1 + 0.6977) + 4(0.99 + 0.9139 + 0.7788) + 2(0.9608 + 0.8521)]$

∴ I = 0.5351

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's 1/3rd rule taking four equal strips and hence deduce an approximate value of π.

Solution:

Let f(x) = $\frac{1}{1+x^2}$. Here a = 0 and b = 1. Take n = 4. ∴ h = $\frac{1-0}{4} = \frac{1}{4}$.

x	x_0= 0	x_1= 1/4	x_2= 1/2	x_3= 3/4	x_4= 1
y= $\frac{1}{1+x^2}$	y_0= 1	y_1= 16/17	y_2= 4/5	y_3= 16/25	y_4=1/2

Using Simpson's one third rule, we have,
I =$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)]$

∴ I = $\frac{1/4}{3}[(1 + 1/2) + 4(16/17 + 16/25) + 2(4/5)]$

∴ I = 0.7854……. (1)

Further, by direct integration, we have, $I = \int_0^1 \frac{dx}{1+x^2} = [tan^{-1}x]_0^1 = tan^{-1}1 - tan^{-1}0 = \frac{\pi}{4}$ … (2)

∴ From (1) and (2), we get, 0.7854 = $\frac{\pi}{4}$. ∴ π = 3.1416.

3. Find the approximate value of $\int_0^{\pi/2} \sqrt{cos\theta}\, d\theta$ by Simpson's one third rule by dividing [0, $\pi/2$] into 6 equal parts. (7 ordinates).

Solution:

Let f(θ) = $\sqrt{cos\theta}$. Here a = 0 and b = $\pi/2$. Take n = 6. ∴ h = $\frac{\pi/2-0}{6} = \pi/12 = 15^0$.

(Use calculator in degree mode) But Take h = $\pi/12$.

θ^0= x	x_0= 0	x_1= 15	x_2= 30	x_3= 45	x_4= 60	x_5= 75	x_6= 90
y= $\sqrt{cos\theta}$	y_0= 1	y_1= 0.9828	y_2= 0.9306	y_3= 0.8409	y_4= 0.7071	y_5= 0.5087	y_6= 0

Using Simpson's one third rule, we have,
I = $\int_0^{\pi/2} \sqrt{cos\theta}\, d\theta = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

$\therefore \quad I = \frac{\pi/12}{3} [(1+0) + 4(0.9828 + 0.8409 + 0.5087) + 2(0.9306 + 0.7071)]$

$\therefore \quad I = 1.1873$

4. The velocity v $(km/hour)$ of a moped which starts from rests, is given at fixed Intervals of time t $(hour)$ as follows:

t :	2	4	6	8	10	12	14	16	18	20
v :	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 hours.

Solution:

Let S be the distance covered by the moped at time t, then $\frac{dS}{dt} = v$. Therefore the distance S covered in 20 hours is given by S = $\int_0^{20} v\ dt$. Here take n = 10, a = 0, b = 20 and h = 2.

$t = x$	$x_0 = 0$	$x_1 = 2$	$x_2 = 4$	$x_3 = 6$	$x_4 = 8$
$v = y$	$y_0 = 0$	$y_1 = 10$	$y_2 = 18$	$y_3 = 25$	$y_4 = 29$

$x_5 = 10$	$x_6 = 12$	$x_7 = 14$	$x_8 = 16$	$x_9 = 18$	$x_{10} = 20$
$y_5 = 32$	$y_6 = 20$	$y_7 = 11$	$y_8 = 5$	$y_9 = 2$	$y_{10} = 0$

Using Simpson's one third rule, we have,

$$S = \int_0^{20} v\ dt = \frac{h}{3} [(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$

$\therefore \quad S = \frac{5}{3} [(0+0) + 4(10 + 25 + 32 + 11 + 2) + 2(18 + 29 + 20 + 5)] = \frac{2}{3}(464)$

$\therefore \quad S = 309.33\ km$

HOMEWORK:

1. **Use Simpson's one third rule with seven ordinates to evaluate $\int_2^8 \frac{dx}{\log_{10} x}$.**

Ans: 9.7203

2. **Then Velocity v of a particle at distance s from a point on its linear path is given by the following table. Estimate the time taken by the particle to traverse the distance 20 meters ,using Simpson's 1/3rd rule.**

S(m)	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
V(m/sec)	16	19	21	22	20	17	13	11	9

3. **Evaluate $\int_0^{\pi/2} \cos x\, dx$ by applying Simpson's one third rule taking eleven**

ordinates. Compare the value with the theoretical value.
Ans: 1.

Simpson's three eighth (3/8th) rule:

If n is a multiple of 3 i.e., n = 3, 6, 9, …… , then

I = $\int_a^b f(x)\,dx$

$= \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})\right].$

Where h = $\frac{b-a}{n}$. This is known as **Simpson's three eighth (3/8th rule.**

Problems:

1. Evaluate $\int_0^3 \frac{dx}{(1+x)^2}$ by Simpson's three eighth rule.

Solution:

Let f(x) = $\frac{1}{(1+x)^2}$. Here a = 0 and b = 3. Take n = 3. ∴ h = $\frac{3-0}{3}$ = 1.

x	x_0= 0	x_1= 1	x_2= 2	x_3= 3
y= $\frac{1}{(1+x)^2}$	y_0= 1	y_1= 1/4	y_2= 1/9	y_3= 1/16

Using Simpson's one third rule, we have,
I =$\int_0^3 \frac{dx}{(1+x)^2} = \frac{3h}{8}\left[(y_0 + y_3) + 3(y_1 + y_2)\right]$

∴ I = $\frac{3}{8}\left[(1 + 1/16) + 3(1/4 + 1/9)\right]$ = 0.8047

2. Evaluate $\int_0^1 \frac{dx}{1+x}$ taking seven ordinates by applying Simpson's three eighth rule.

Hence deduce the value of log 2.

Solution:

Let f(x) = $\frac{1}{1+x}$. Here a = 0 and b = 1. Take n = 6. ∴ h = $\frac{1-0}{6}$ = 1/6.

x	x_0= 0	x_1= 1/6	x_2= 2/6 = 1/3	x_3= 3/6 = 1/2	x_0= 4/6 = 2/3	x_0= 5/6	x_0= 6/6 = 1
y = $\frac{1}{1+x}$	y_0= 1	y_1= 6/7	y_2= 3/4	y_0= 2/3	y_0= 3/5	y_0= 6/11	y_0= 1/2

Using Simpson's three eighth rule, we have,

$I = \int_0^1 \frac{dx}{1+x} = \frac{3h}{8}\,[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$

$\therefore$ $I = \frac{1}{16}\,[(1 + 1/2) + 3(6/7 + 3/4 + 3/5 + 6/11) + 2(2/3)]$ $\therefore$ I = 0.6932 ….. (1)

Further, by direct integration, we have, $I = \int_0^1 \frac{dx}{1+x} = [\log(1+x)]_0^1 = \log 2$ ……. (2)

$\therefore$ From (1) and (2), we get, log 2 = = 0.6932.

3. By using Simpson's three eighth rule evaluate $\int_0^{\pi/2} e^{\sin\theta}\, d\theta$ by dividing the interval into three equal parts.

Solution:

Let $f(x) = e^{\sin\theta}$. Here a = 0 and $b = \pi/2$. Take n = 3. $\therefore$ $h = \frac{\pi/2 - 0}{3} = \pi/6$.

$\theta = x$	$x_0 = 0$	$x_1 = \pi/6$	$x_2 = \pi/3$	$x_3 = \pi/2$
$y = e^{\sin\theta}$	$y_0 = 1$	$y_1 = 1.6487$	$y_2 = 2.3774$	$y_3 = 2.7183$

Using Simpson's three eighth rule, we have,

$I = \int_0^{\pi/2} e^{\sin\theta} = \frac{3h}{8}\,[(y_0 + y_3) + 3(y_1 + y_2)]$

$\therefore$ $I = \frac{\pi}{16}\,[(1 + 2.7183) + 3(1.6487 + 2.3774)]$

$\therefore$ $I = 0.9873\pi = 3.1017$.

4. Compute the value of $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\, dx$ using Simpson's $\frac{3}{8}$th rule.

Solution:

Let $f(x) = \sin x - \log x + e^x$. Here a = 0.2 and b = 1.4. Take n = 6. $\therefore$ $h = \frac{1.4 - 0.2}{6} = 0.2$.

x	$x_0 = 0.2$	$x_1 = 0.4$	$x_2 = 0.6$	$x_3 = 0.8$	$x_4 = 1.0$	$x_5 = 1.2$	$x_6 = 1.4$
y = f(x)	$y_0 = 3.0295$	$y_1 = 2.7975$	$y_2 = 2.8976$	$y_3 = 3.1660$	$y_4 = 3.5597$	$y_5 = 4.0698$	$y_6 = 4.4042$

Using Simpson's three eighth rule, we have,

$I = \int_{0.2}^{1.4} (\sin x - \log x + e^x)\, dx = \frac{3h}{8}\,[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$

$\therefore$ $I = \frac{3(0.2)}{8}\,[(3.0295 + 4.4042) + 2(3.1660) + 3(2.7975 + 2.8976 + 3.5597 + 4.0698)]$

$\therefore$ $I = 4.053$

HOMEWORK:

1. Use Simpson's three eighth rule to obtain the approximate value $\int_0^{0.3}(1-8x^3)^{1/2}\,dx$

by considering 3 equal intervals. Ans: 0.2916

2. Use Simpson's $\frac{3}{8}$ th rule to evaluate $\int_1^4 e^{1/x}\,dx.$ Ans: 4.9257

3. A rocket is launched from the ground .Its acceleration a is registered during the first one minute and is given below. Apply the Simpson's 3/8th rule to find the velocity of the rocket at the first minute.

Time(in seconds)	0	10	20	30	40	50	60
Acceleration(cm/sec^2	30	31.63	33.34	35.47	37.75	40.33	43.25