NAGARJUNA COLLEGE OF ENGINEERING AND TECHNOLOGY
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NAGARJUNA
COLLEGE OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF MATHEMATICS

ADVANCED CALCULUS AND NUMERICAL METHODS

(COURSE CODE 22MATS21/22MATC21/22MATE21)

CLASS NOTES FOR II SEM B.E.

MODULE-1

VECTOR CALCULUS

SYLLABUS:

Vector Differentiation: Scalar and vector fields. Gradient, directional derivative, curl and divergence -physical interpretation, solenoidal and irrotational vector fields. Problems.
Vector Integration: Line integrals, Surface integrals. Applications to work done by a force and flux.Statement of Green's theorem and Stoke's theorem. Problems.

VECTOR DIFFERENTIATION:

If a vector $\vec{r}$ varies continuously as a scalar variable t changes, then $\vec{r}$ is called a function of t and is written as $\vec{r} = \vec{F}(t)$.

The derivative of a vector function $\vec{r} = \vec{F}(t)$ is denoted by $\frac{d\vec{r}}{dt}$ or $\frac{d\vec{F}}{dt}$ or $\vec{F}'(t)$ and is defined by $\frac{d\vec{F}}{dt} = \lim_{\delta t \to 0} \frac{\vec{F}(t+\delta t) - \vec{F}(t)}{\delta t}$.

Scalar point function:

Definition:

If to each point $P(x, y, z)$ in the region R of a space with the position vector $\vec{r}$ there exists a definite scalar $\emptyset(x, y, z)$, then $\emptyset(x, y, z)$ is called the scalar point function in R and the region R is called the scalar field.

Example: (i) $\emptyset = xyz$ and (ii) $\psi = x^2 + y^2 + z^2$ are scalar point functions.

Vector point function:

Definition:

If to each point $P(x, y, z)$ in the region R of a space with the position vector $\vec{r}$ there exists a definite vector $\vec{F}(x, y, z)$, then $\vec{F}(x, y, z)$ is called the vector point function in R and the region R is called the vector field.

Example: (i) $\vec{F} = (x + y)\hat{i} + xy\,\hat{j} + z\,\hat{k}$ and

(ii) $\vec{A} = (x - y^2)\hat{i} + x^2z\hat{j} + (x + y)\hat{k}$ are vector point functions.

The vector differential operator:

The vector differential operator denoted by ∇(read as del or nabla) and is defined by

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \sum \frac{\partial}{\partial x}\hat{i}\,.$$

Gradient of a scalar field:

If $\emptyset = \emptyset(x, y, z)$ be any scalar point function then gradient of $\emptyset$ is defined by

$$\text{grad}\,\emptyset = \nabla\emptyset = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\emptyset = \frac{\partial\emptyset}{\partial x}\hat{i} + \frac{\partial\emptyset}{\partial y}\hat{j} + \frac{\partial\emptyset}{\partial z}\hat{k} = \sum\frac{\partial\emptyset}{\partial x}\hat{i}.$$

Here $\nabla\emptyset$ is a vector quantity.

Physical Interpretation of Gradient:

Gradient ∇F tells us that in which direction change in the field (F) is maximum.

Geometrical meaning:

Let $\emptyset(x, y, z) = c$ be the equation of a surface, then $\nabla\emptyset$ is the normal vector at a point $P(x, y, z)$ to the surface $\emptyset(x, y, z) = c$. Therefore the unit normal vector to the surface $\emptyset(x, y, z) = c$ is given by $\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|}$

Note:

(i) We know that the angle between the two surfaces is defined as the angle between their normal. Therefore the angle θ between two surfaces $\emptyset_1(x, y, z) = c_1$ and $\emptyset_2(x, y, z) = c_2$ is equal to the angle between their normal $\nabla\emptyset_1$ and $\nabla\emptyset_2$ and is given by

$$\cos\theta = \frac{\nabla\emptyset_1 \cdot \nabla\emptyset_2}{|\nabla\emptyset_1||\nabla\emptyset_2|}.$$

(ii) If $\theta = \frac{\pi}{2}$ or 90°, then the surfaces are said to be intersect each other orthogonally.

i.e., If $\nabla\emptyset_1 \cdot \nabla\emptyset_2 = 0$, then the surfaces are said to be intersect each other orthogonally.

Problems:

1. Find ∇f when $f = \log(x^2 + y^2 + z^2)$

Solution:

Let $f = \log(x^2 + y^2 + z^2)$.................. (1)

We have $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$..........(2)

Differentiating equation (1) partially w. r. t. x, y and z we get,

$\frac{\partial f}{\partial x} = \frac{2x}{(x^2+y^2+z^2)}$, $\frac{\partial f}{\partial y} = \frac{2y}{(x^2+y^2+z^2)}$ and $\frac{\partial f}{\partial z} = \frac{2z}{(x^2+y^2+z^2)}$. **Substituting in (2), we get,**

$\nabla f = \frac{2x}{(x^2+y^2+z^2)}\hat{\imath} + \frac{2y}{(x^2+y^2+z^2)}\hat{\jmath} + \frac{2z}{(x^2+y^2+z^2)}\hat{k}$. **Thus** $\nabla f = \frac{2(x\,\hat{\imath}+y\,\hat{\jmath}+z\hat{k})}{(x^2+y^2+z^2)}$.

2. If $\emptyset = x^2y^3z^4$, find $\nabla\emptyset, |\nabla\emptyset|$ at (1,-1,1)

Solution:

Let $\emptyset = x^2y^3z^4$ **……………… (1)**

We have $\nabla\emptyset = \frac{\partial\emptyset}{\partial x}\hat{\imath} + \frac{\partial\emptyset}{\partial y}\hat{\jmath} + \frac{\partial\emptyset}{\partial z}\hat{k}$**……….(2)**

Differentiating equation (1) partially w. r. t. x, y and z we get,

$\frac{\partial\emptyset}{\partial x} = 2xy^3z^4$ $\frac{\partial\emptyset}{\partial y} = 3x^2y^2z^4$ and $\frac{\partial\emptyset}{\partial z} = 4x^2y^3z^3$. **Substituting in (2), we get,**

$\nabla\emptyset = 2xy^3z^4\ \hat{\imath} + 3x^2y^2z^4\,\hat{\jmath} + 4x^2y^3z^3\,\hat{k}$.

At (1,-1,1)

$\nabla\emptyset = -2\,\hat{\imath} + 3\,\hat{\jmath} - 4\hat{k}$, *and* $|\nabla\emptyset| = \sqrt{(-2)^2 + 3^2 + (-4)^2} = \sqrt{29}$

3. If $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\hat{k}$ and $r = |\vec{r}|$ then prove that $\nabla(r^n) = nr^{n-2}\vec{r}$.

Solution:

Given $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\hat{k} = \sum x\,\hat{\imath}$ **… … (1)** $\therefore$ $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$\therefore$ $r^2 = x^2 + y^2 + z^2$, **Differentiating partially w. r. t. x we get** $2r\frac{\partial r}{\partial x} = 2x$ $\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$,

Similarly, we get $\frac{\partial r}{\partial y} = \frac{y}{r}$ **and** $\frac{\partial r}{\partial z} = \frac{z}{r}$. **Using** $\nabla\emptyset = \sum\frac{\partial\emptyset}{\partial x}\hat{\imath}$, **we get,**

$\nabla(r^n) = \sum\frac{\partial r^n}{\partial x}\hat{\imath} = \sum nr^{n-1}\frac{\partial r}{\partial x}\hat{\imath} = n\sum r^{n-1}\ \frac{x}{r}\hat{\imath} = n\sum r^{n-2}\ x\,\hat{\imath} = nr^{n-2}\sum x\,\hat{\imath}$.

$\therefore$ $\nabla(r^n) = nr^{n-2}\vec{r}$

4. Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$.

Solution:

Let $\emptyset = xy^3z^2 - 4 = 0$ **……… (1)**

Differentiating equation (1) partially w. r. t. x, y and z we get,

$$\frac{\partial \emptyset}{\partial x} = y^3z^2, \quad \frac{\partial \emptyset}{\partial y} = 3xy^2z^2, \quad \frac{\partial \emptyset}{\partial z} = 2xy^3z.$$

We have $\nabla\emptyset = \frac{\partial \emptyset}{\partial x}\hat{i} + \frac{\partial \emptyset}{\partial y}\hat{j} + \frac{\partial \emptyset}{\partial z}\hat{k}$

$\therefore \nabla\emptyset = y^3z^2\hat{i} + 3xy^2z^2\hat{j} + 2xy^3z\hat{k} \quad \therefore \nabla\emptyset_{(-1,-1,2)} = -4\,\hat{i} - 12\hat{j} + 4\,\hat{k}$

$\therefore |\nabla\emptyset| = \sqrt{(-4)^2 + (-12)^2 + 4^2} = \sqrt{176} = \sqrt{16 * 11} = 4\sqrt{11}$

$\therefore$ **The unit vector normal is** $\widehat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{-4\,\hat{i}-12\hat{j}+4\,\hat{k}}{4\sqrt{11}} = \frac{-\,\hat{i}-3\hat{j}+\hat{k}}{\sqrt{11}}$.

5. Find the angle between the tangent planes to the surfaces $x\log z = y^2 - 1$ **and** $x^2y = 2 - z$ **at the point** $(1, 1, 1)$

Solution:

Let $\emptyset_1 = x\log z - y^2 + 1 = 0$ **.....(1) and** $\emptyset_2 = x^2y - 2 + z = 0$ **.....(2)**

$\therefore \frac{\partial \emptyset_1}{\partial x} = \log z, \quad \frac{\partial \emptyset_1}{\partial y} = -2y, \quad \frac{\partial \emptyset_1}{\partial z} = \frac{x}{z}, \quad \frac{\partial \emptyset_2}{\partial x} = 2xy, \quad \frac{\partial \emptyset_2}{\partial y} = x^2, \quad \frac{\partial \emptyset_2}{\partial z} = 1.$

We have $\nabla\emptyset = \frac{\partial \emptyset}{\partial x}\hat{i} + \frac{\partial \emptyset}{\partial y}\hat{j} + \frac{\partial \emptyset}{\partial z}\hat{k}$

$\therefore \nabla\emptyset_1 = \log z\,\hat{i} - 2y\hat{j} + \frac{x}{z}\hat{k}$ **and** $\nabla\emptyset_2 = 2xy\,\hat{i} + x^2\hat{j} + \widehat{k}$

$\therefore$ **At** $(1, 1, 1)$, $\nabla\emptyset_1 = 0\,\hat{i} - 2\hat{j} + \hat{k}$ **and** $\nabla\emptyset_2 = 2\,\hat{i} + \hat{j} + \widehat{k}$

$\therefore |\nabla\emptyset_1| = \sqrt{0^2 + (-2)^2 + 1^2} = \sqrt{5}$ **and** $|\nabla\emptyset_2| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

Let θ **be the angle between the tangent planes of the two surfaces, then**

we have, $\cos\theta = \frac{\nabla\emptyset_1 . \nabla\emptyset_2}{|\nabla\emptyset_1||\nabla\emptyset_2|} \quad \therefore \cos\theta = \frac{(0\,\hat{i}-2\hat{j}+\hat{k}).(2\,\hat{i}+\hat{j}+\widehat{k})}{\sqrt{5}\sqrt{6}} = \frac{0-2+1}{\sqrt{30}} = \frac{-1}{\sqrt{30}}$

$\therefore \theta = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right).$

6. Find the angle between the directions of the normal to the surface $x^2yz = 1$ **at the points** $(-1, 1, 1)$ ***and*** $(1, -1, -1)$.

Solution:

Let $\emptyset = x^2yz - 1$

We have $\nabla\emptyset = \frac{\partial\emptyset}{\partial x}\hat{\imath} + \frac{\partial\emptyset}{\partial y}\hat{\jmath} + \frac{\partial\emptyset}{\partial z}\hat{k}$

$$\nabla\emptyset = 2xyz\hat{\imath} + x^2z\,\hat{\jmath} + x^2y\,\hat{k}$$

i) At $(-1,1,1)$ $\nabla\emptyset = -2\hat{\imath} + \hat{\jmath} + \hat{k}$

$|\nabla\emptyset| = \sqrt{6}$

$$\widehat{n_1} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{-2\hat{\imath} + \hat{\jmath} + \hat{k}}{\sqrt{6}}$$

ii) At $(1,-1,-1)$ $\nabla\emptyset = 2\hat{\imath} - \hat{\jmath} - \hat{k}$

$|\nabla\emptyset| = \sqrt{6}$

$$\widehat{n_2} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{2\hat{\imath} - \hat{\jmath} - \hat{k}}{\sqrt{6}}$$

$\cos\theta = \widehat{n_1} \cdot \widehat{n_2} = \frac{-2\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{6}} \cdot \frac{2\hat{\imath}-\hat{\jmath}-\hat{k}}{\sqrt{6}} = -1$

$\theta = 180^\circ = \pi$

7. Find the values of a and b such that the surfaces $5x^2 - 2yz - 9z + 9 = 0$ may cut the surface $ax^2 + by^3 = 4$ orthogonally at the point $(1,-1,\ 2)$.

Solution:

Let $\emptyset_1 = 5x^2 - 2yz - 9z + 9 = 0$ ……(1) and $\emptyset_2 = ax^2 + by^3 - 4 = 0$……(2)

$\therefore\ \frac{\partial\emptyset_1}{\partial x} = 10x,\ \frac{\partial\emptyset_1}{\partial y} = -2z,\ \frac{\partial\emptyset_1}{\partial z} = -2y - 9,\ \frac{\partial\emptyset_2}{\partial x} = 2ax,\ \frac{\partial\emptyset_2}{\partial y} = 3by^2,\ \frac{\partial\emptyset_2}{\partial z} = 0.$

We have $\nabla\emptyset = \frac{\partial\emptyset}{\partial x}\hat{\imath} + \frac{\partial\emptyset}{\partial y}\hat{\jmath} + \frac{\partial\emptyset}{\partial z}\hat{k}$

$\nabla\emptyset_1 = 10x\,\hat{\imath} - 2z\hat{\jmath} + (-2y - 9)\hat{k}$ and $\nabla\emptyset_2 = 2ax\,\hat{\imath} + 3by^2\hat{\jmath} + 0\,\hat{k}$

At $(1,-1,\ 2)$, $\nabla\emptyset_1 = 10\,\hat{\imath} - 4\hat{\jmath} - 7\,\hat{k}$ and $\nabla\emptyset_2 = 2a\,\hat{\imath} + 3b\,\hat{\jmath} + 0\,\hat{k}$

If two surfaces cut orthogonally, then we have, $\nabla\emptyset_1 . \nabla\emptyset_2 = 0$

$\therefore \left(10\,\hat{\imath} - 4\hat{\jmath} - 7\,\hat{k}\right) . \left(2a\,\hat{\imath} + 3b\,\hat{\jmath} + 0\,\hat{k}\right) = 0$

$\therefore\ 20a - 12b = 0 \quad \therefore 5a - 3b = 0$……(3)

The point $(1,-1,\ 2)$ lies on the equation (2). $\therefore$ we get $a - b = 4$……. (4)

Solving equations (3) and (4), we get $a = -6$ and $b = -10$.

HOME WORK:

1. If $f = 3x^2y - y^3z^2$ find ∇f and $|\nabla f|$ at $(1, -2, -1)$
2. Find a unit vector normal to the surface $xy + yz + zx = c$ at the point $(-1, 2, 3)$.
3. Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$.
4. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $P(2, -1, 2)$.
5. Find the constants a and b such that the surface $ax^2 - byz = (a + 2)x$ will be orthogonal to the Surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$.

Directional derivatives:

Definition:

If $\emptyset(x, y, z)$ is any scalar point function and $\vec{d}$ is a given direction, then $\nabla\emptyset.\hat{n}$, where $\hat{n} = \frac{\vec{d}}{|\vec{d}|}$, is called the directional derivative of $\emptyset$ along $\vec{d}$.

Note:

The directional derivative of a scalar function $\emptyset$ at any point is maximum along $\nabla\emptyset$ and its maximum value is equal to $|\nabla\emptyset|$.

Problems:

1. Find the directional derivative of $\emptyset = x^2yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction of the vector $2\hat{\imath} - \hat{\jmath} - 2\hat{k}$.

Solution:

Given $\emptyset = x^2yz + 4xz^2 \quad \therefore \ \frac{\partial\emptyset}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial\emptyset}{\partial y} = x^2z, \quad \frac{\partial\emptyset}{\partial z} = x^2y + 8xz.$

We have $\nabla\emptyset = \frac{\partial\emptyset}{\partial x}\hat{\imath} + \frac{\partial\emptyset}{\partial y}\hat{\jmath} + \frac{\partial\emptyset}{\partial z}\hat{k}$

$\therefore \nabla\emptyset = (2xyz + 4z^2)\hat{\imath} + (x^2z)\hat{\jmath} + (x^2y + 8xz)\hat{k} \qquad \therefore \ \nabla\emptyset_{(1,\ -2,\ 1)} = 0\,\hat{\imath} + \hat{\jmath} + 6\,\hat{k}$

Let $\vec{d} = 2\hat{\imath} - \hat{\jmath} - 2\hat{k}. \quad \therefore \ |\vec{d}| = \sqrt{(2)^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$

$\therefore \ \hat{n} = \frac{\vec{d}}{|\vec{d}|} = \frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{3}$

$\therefore$ The directional derivative of $\emptyset$ in the direction of $\vec{d}$ is

$$\nabla\emptyset.\,\hat{n} = \left(0\,\hat{\imath} + \hat{\jmath} + 6\,\hat{k}\right).\left(\frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{3}\right) = \frac{0 - 1 - 12}{3} = \frac{-13}{3}.$$

2. Find the directional derivative of $\emptyset = xy^2 + yz^3$ at the point $(2, -1,\ 1)$ in the direction of the of the normal to the surface $x\log z - y^2 = -4$ at $(-1, 2, 1)$

Solution:

Given $\emptyset = xy^2 + yz^3 \quad \therefore \ \frac{\partial\emptyset}{\partial x} = y^2, \quad \frac{\partial\emptyset}{\partial y} = 2xy + z^3, \quad \frac{\partial\emptyset}{\partial z} = 3yz^2.$

We have $\nabla\emptyset = \frac{\partial\emptyset}{\partial x}\hat{\imath} + \frac{\partial\emptyset}{\partial y}\hat{\jmath} + \frac{\partial\emptyset}{\partial z}\hat{k}$

$\therefore \ \nabla\emptyset = y^2\hat{\imath} + (2xy + z^3)\hat{\jmath} + 3yz^2\hat{k} \qquad \therefore \ \nabla\emptyset_{(2,-1,\ 1)} = \hat{\imath} - 3\hat{\jmath} - 3\hat{k}$

Let the given surface be $\Psi = x\log z - y^2 + 4 = 0 \quad \therefore \ \frac{\partial\Psi}{\partial x} = \log z, \quad \frac{\partial\Psi}{\partial y} = -2y, \quad \frac{\partial\Psi}{\partial z} = \frac{x}{z}$

We have $\nabla\Psi = \frac{\partial\Psi}{\partial x}\hat{\imath} + \frac{\partial\Psi}{\partial y}\hat{\jmath} + \frac{\partial\Psi}{\partial z}\hat{k}$

$\therefore \ \nabla\Psi = \log z\,\hat{\imath} - 2y\hat{\jmath} + \frac{x}{z}\hat{k} \qquad \therefore \ \nabla\Psi_{(-1,\ 2,\ 1)} = 0\,\hat{\imath} - 4\hat{\jmath} - \hat{k}$

$\therefore$ The normal to the $surface\ \Psi\ is\ \nabla\Psi = -4\hat{\jmath} - \hat{k}.$

$\therefore$ Let $\vec{d} = \nabla\Psi = -4\hat{\jmath} - \hat{k}. \quad \therefore \ |\vec{d}| = \sqrt{(-4)^2 + (-1)^2} = \sqrt{17} \quad \therefore \hat{n} = \frac{\vec{d}}{|\vec{d}|} = \frac{-4\hat{\jmath} - \hat{k}}{\sqrt{17}}$

$\therefore$ The directional derivative of $\emptyset$ in the direction of $\vec{d}$ is

$$\nabla\emptyset.\,\hat{n} = \left(\hat{\imath} - 3\hat{\jmath} - 3\hat{k}\right).\left(\frac{-4\hat{\jmath} - \hat{k}}{\sqrt{17}}\right) = \frac{0 + 12 + 3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$$

3. Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at the point $P(1,\ 2,\ 3)$ in the

direction of the line PQ where Q is the point (5, 0, 4). Also calculate the magnitude of the maximum directional derivative.

Solution:

Given $f = x^2 - y^2 + 2z^2 \quad \therefore \frac{\partial f}{\partial x} = 2x, \ \frac{\partial f}{\partial y} = -2y, \ \frac{\partial f}{\partial z} = 4z$

We have $\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$

$\therefore \nabla f = 2x\,\hat{\imath} - 2y\,\hat{\jmath} + 4z\,\hat{k} \quad \therefore \nabla f_{(1,\ 2,\ 3)} = 2 - 4\hat{\jmath} + 12\hat{k}$

Given $P = (1,\ 2,\ 3)$ **and** $Q = (5,\ 0,\ 4)$

$\therefore \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (5\,\hat{\imath} + 0\,\hat{\jmath} + 4\,\hat{k}) - (\hat{\imath} + 2\hat{\jmath} + 3\,\hat{k}) = 4\,\hat{\imath} - 2\hat{\jmath} + \hat{k}$

Now take $\vec{d} = \overrightarrow{PQ} = 4\,\hat{\imath} - 2\hat{\jmath} + \hat{k} \quad \therefore \ |\vec{d}| = \sqrt{(4)^2 + (-2)^2 + (1)^2} = \sqrt{21}$

$\therefore \ \hat{n} = \frac{\vec{d}}{|\vec{d}|} = \frac{4\,\hat{\imath} - 2\hat{\jmath} + \hat{k}}{\sqrt{21}}$

$\therefore$ **The directional derivative of f along** $\vec{d} = \overrightarrow{PQ}$ **is**

$\nabla f.\,\hat{n} = (2\,\hat{\imath} - 4\,\hat{\jmath} + 12\,\hat{k}).\frac{(4\,\hat{\imath} - 2\hat{\jmath} + \hat{k})}{\sqrt{21}} = \frac{(8+8+12)}{\sqrt{21}} = \frac{28}{\sqrt{21}}$

$\therefore$ **The magnitude of the maximum directional derivative is**

$|\nabla f| = \sqrt{(2)^2 + (-4)^2 + (12)^2} = \sqrt{164}$

HOME WORK:

1. Find the directional derivative of $f = xy^3 + yz^3$ **at the point** $(2, -1,\ 1)$ **in the direction of the vector** $\hat{\imath} + 2\,\hat{\jmath} + 2\,\hat{k}$.

Divergence of a vector function:

Definition:

If $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ **is a vector function differentiable at each point** (x, y, z)**,**

then the divergence of $\vec{F}$ **is denoted by** $div\vec{F}$ *or* $\nabla \cdot \vec{F}$ **and is defined by**

$$div\vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}\right) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

$\therefore \ div\vec{F} = \sum \frac{\partial F_1}{\partial x}$. **Here** $div\vec{F}$ **is a scalar quantity.**

Curl of a vector function:

Definition:

If $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ is any vector function differentiable at each point (x, y, z),

then curl of $\vec{F}$ is denoted by curl$\vec{F} = \nabla X\vec{F}$ and is defined by

$$\text{curl}\vec{F} = \nabla X\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$\therefore$ curl$\vec{F} = \sum\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath}$. Here curl$\vec{F}$ is a vector quantity.

Physical Interpretation of Divergence and Curl:

The divergence of a vector field represents the out flow rate from a point.However the curl of a vector field represents the rotation at a point.

Problems:

1. Evaluate divergence of $2x^2z\,\hat{\imath} - xy^2z\,\hat{\jmath} + 3yz^2\hat{k}$ at the point $(1, 1, 1)$.

Solution:

Let $\vec{F} = 2x^2z\,\hat{\imath} - xy^2z\,\hat{\jmath} + 3yz^2\hat{k}$. i.e., $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$

Where $F_1 = 2x^2z$, $F_2 = -xy^2z$, $F_3 = 3yz^2$.

$$div\vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(2x^2z) + \frac{\partial}{\partial y}(-xy^2z) + \frac{\partial}{\partial z}(3yz^2)$$

$\therefore$ $div\vec{F} = 4xz - 2xyz + 6yz$. $\therefore$ $div\vec{F}_{(1,1,1)} = 4 - 2 + 6 = 8$.

2. Evaluate curl $\vec{F}$ at the point $(1, 2, 3)$ given $\vec{F} = x^2yz\,\hat{\imath} + xy^2z\,\hat{\jmath} + xyz^2\hat{k}$.

Solution:

Given $\vec{F} = x^2yz\,\hat{\imath} + xy^2z\,\hat{\jmath} + xyz^2\hat{k}$. i.e., $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$

Where $F_1 = x^2yz$, $F_2 = xy^2z$, $F_3 = xyz^2$.

$$\therefore curl\vec{F} = \nabla X\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(xyz^2) - \frac{\partial}{\partial z}(xy^2z)\right)\hat{\imath} - \left(\frac{\partial}{\partial x}(xyz^2) - \frac{\partial}{\partial z}(x^2yz)\right)\hat{\jmath} + \left(\frac{\partial}{\partial x}(xy^2z) - \frac{\partial}{\partial y}(x^2yz)\right)\hat{k}$$

$$= (xz^2 - xy^2)\hat{\imath} - (yz^2 - x^2y)\hat{\jmath} + (y^2z - x^2z)\hat{k}$$

$$\therefore curl\vec{F}_{(1,2,3)} = 5\,\hat{\imath} - 16\hat{\jmath} + 9\,\hat{k}.$$

3. Evaluate div $\vec{F}$ and curl $\vec{F}$ at the point $(1, 2, 3)$ where

$$\vec{F} = grad(x^3y + y^3z + z^3x - x^2y^2z^2)$$

Solution:

Let $\emptyset = x^3y + y^3z + z^3x - x^2y^2z^2$, **then** $\vec{F} = grad\emptyset = \frac{\partial\emptyset}{\partial x}\hat{\imath} + \frac{\partial\emptyset}{\partial y}\hat{\jmath} + \frac{\partial\emptyset}{\partial z}\hat{k}$

$$\therefore \quad \vec{F} = (3x^2y + z^3 - 2xy^2z^2)\hat{\imath} + (x^3 + 3y^2z - 2yx^2z^2)\hat{\jmath} + (y^3 + 3z^2x - 2zx^2y^2)\hat{k}$$

i.e., $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$. **Where** $F_1 = 3x^2y + z^3 - 2xy^2z^2$,

$F_2 = x^3 + 3y^2z - 2yx^2z^2$, $F_3 = y^3 + 3z^2x - 2zx^2y^2$.

Now $div\vec{F} = \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$= \frac{\partial}{\partial x}(3x^2y + z^3 - 2xy^2z^2) + \frac{\partial}{\partial y}(x^3 + 3y^2z - 2yx^2z^2) + \frac{\partial}{\partial z}(y^3 + 3z^2x - 2zx^2y^2)$$

$$= (6xy - 2y^2z^2) + (6yz - 2x^2z^2) + (6xz - 2x^2y^2)$$

$$\therefore div\,\vec{F}_{(1,\ 2,\ 3)} = 12 - 72 + 36 - 18 + 18 - 8 = -32.$$

Now,

$$Curl\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2y + z^3 - 2xy^2z^2) & (x^3 + 3y^2z - 2yx^2z^2) & (y^3 + 3z^2x - 2zx^2y^2) \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(y^3 + 3z^2x - 2zx^2y^2) - \frac{\partial}{\partial z}(x^3 + 3y^2z - 2yx^2z^2)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}(y^3+3z^2x-2zx^2y^2)-\frac{\partial}{\partial z}(3x^2y+z^3-2xy^2z^2)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(x^3+3y^2z-2yx^2z^2\ )-\frac{\partial}{\partial y}(3x^2y+z^3-2xy^2z^2)\right]$$

$$=[(3y^2-4zx^2y)-(3y^2-4yx^2z)]\hat{\imath}-[(3z^2-4xzy^2)-(3z^2-4xzy^2)]\hat{\jmath}$$

$$+[(3x^2-4yxz^2)-(3x^2-4xyz^2)]\hat{k}$$

$\therefore\quad Curl\,\vec{F}_{(1,2,3)}=0.$

4. Find curl(curl) of $\vec{A}=x^2y\,\hat{\imath}-2xz\,\hat{\jmath}\,+2yz\hat{k}$ at the point $(1,0,2)$

Solution:

Given $\vec{A}=x^2y\,\hat{\imath}-2xz\,\hat{\jmath}\,+2yz\hat{k}$ i.e., $\vec{A}=A_1\hat{\imath}+A_2\hat{\jmath}\,+A_3\hat{k}$.

Where $A_1=x^2y,\quad A_2=-2xz,\quad A_3=2yz.$

We have, $\text{Curl}\vec{A}=\nabla X\vec{A}=\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ A_1&A_2&A_3\end{vmatrix}=\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ x^2y&-2xz&2yz\end{vmatrix}$

$\therefore\quad \text{Curl}\vec{A}=\hat{\imath}\left(\frac{\partial}{\partial y}(2yz)-\frac{\partial}{\partial z}(-2xz)\right)-\hat{\jmath}\left(\frac{\partial}{\partial x}(2yz)-\frac{\partial}{\partial z}(x^2y)\right)+\hat{k}\left(\frac{\partial}{\partial x}(-2xz)-\frac{\partial}{\partial y}(x^2y)\right)$

$\therefore\quad curl\vec{A}=(2z+2x)\hat{\imath}\,-(0-0)\hat{\jmath}\,+(-2z-x^2)\hat{k}$

$\therefore\quad curl\left(curl\vec{A}\right)=\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ 2z+2x&0&-2z-x^2\end{vmatrix}$

$\therefore curl\left(curl\vec{A}\right)=\hat{\imath}(0-0)-\hat{\jmath}(-2x-2)+\hat{k}\,(0-0)=2(x+1)\,\hat{\jmath}$

$\therefore\ curl(curl\vec{A})_{(1,0,2)}=4\hat{\jmath}$

HOME WORK:

1. Find the divergence and curl of the vector $\vec{V}=(xyz)\hat{\imath}+(3x^2y)\hat{\jmath}+(xz^2-y^2z)\hat{k}$ at the point $(2,-1,1)$.

2. Evaluate div $\vec{F}$ and curl $\vec{F}$ at the point $(1,2,3)$ given $\vec{F} = 3x^2\,\hat{\imath} + 5xy^2\,\hat{\jmath} + 5xyz^3\,\hat{k}$.

3. If $\vec{F} = (x+y+1)i + j - (x+y)k$ show that $\vec{F}.curl\vec{F} = 0$.

4. Evaluate curl of $2x^2z\,\hat{\imath} - xy^2z\,\hat{\jmath} + 3yz^2\,\hat{k}$ at the point $(1,1,1)$.

5. Evaluate div $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = grad[x^3 + y^3 + z^3 - 3xyz]$.

6. Find curl(grand Ø), given Ø = $x^2 + y^2 - z$.

7. If $\vec{F} = (x+y+z)\hat{\imath} + \hat{\jmath} - (x+y)\hat{k}$ then show that $\vec{F}.curl\,\vec{F} = 2 - z$

Solenoidal vectors:

Definition:

A vector point function $\vec{F}$ is said to be solenoidal vector point function if $div\vec{F} = 0$

Irrotational vector field or conservative force field:

Definition:

A vector field $\vec{F}$ is said to be Irrotational vector field if curl$\vec{F} = 0$

Irrotational vector field is also known as conservative force field or potential field.

Problems:

1. Show that $\vec{F} = 3y^4z^2\hat{\imath} + 4x^3z^2\hat{\jmath} + 3x^2y^2\hat{k}$ is solenoidal.

Solution:

Given, $\vec{F} = 3y^4z^2\hat{\imath} + 4x^3z^2\hat{\jmath} + 3x^2y^2\hat{k}$

i.e., $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$. **Where** $F_1 = 3y^4z^2$, $F_2 = 4x^3z^2$, $F_3 = 3x^2y^2$.

We have $div\vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$\therefore div\vec{F} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(3x^2y^2) = 0 + 0 + 0 = 0.$

$\therefore$ **Hence $\vec{F}$ is solenoidal.**

2. Prove that $\vec{A} = (6xy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3xz^2 - y)\hat{k}$ is irrotational and find a scalar function $f(x,y,z)$ such that $\vec{A} = \nabla f$.

Solution:

Given $\vec{A} = (6xy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3xz^2 - y)\hat{k}$. i.e., $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$.

Where $A_1 = 6xy + z^3$, $A_2 = 3x^2 - z$, $A_3 = 3xz^2 - y$.

We have $\nabla \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$

$\therefore \quad \nabla \times \vec{A} = \hat{\imath}(-1 + 1) - \hat{\jmath}(3z^2 - 3z^2) + \hat{k}(6x - 6x) = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(0) = 0.$

$\therefore \quad \vec{A}\, is\ irrotational.$

Now given $\vec{A} = \nabla f$

$\therefore \; (6xy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3xz^2 - y)\hat{k} = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}.$

Comparing on both sides we get,

$\frac{\partial f}{\partial x} = 6xy + z^3$……(1), $\quad \frac{\partial f}{\partial y} = 3x^2 - z$……(2), $\quad \frac{\partial f}{\partial z} = 3xz^2 - y$……(3).

Integrating (1) w. r. t. x by treating y and z as constants we get,

$f = 3x^2y + xz^3 + f_1(y, z)$……..(4)

Integrating (2) w. r. t. y by treating x and z as constants we get,

$f = 3x^2y - yz + f_2(x, z)$……..(5)

Integrating (3) w. r. t. y by treating x and y as constants we get,

$f = xz^3 - yz + f_3(x, y)$……….(6)

Using (4), (5) and (6), we write

$f = 3x^2y + xz^3 - yz$

3. If $\vec{A} = (x + y + az)\hat{\imath} + (bx + 2y - z)\hat{\jmath} + (x + cy + 2z)\hat{k}$ find a and b such that curl $\vec{A} = 0$ (i.e., $\vec{A}$ is irrotational).

Solution:

Given $\vec{A} = (x + y + az)\hat{\imath} + (bx + 2y - z)\hat{\jmath} + (x + cy + 2z)\hat{k}$

i.e., $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$.

Where $A_1 = x + y + az, \quad A_2 = bx + 2y - z, \quad A_3 = x + cy + 2z.$

Given $curl\,\vec{A} = 0.$ i.e., $\vec{A}$ is irrotational. $\therefore \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = 0$

$$\therefore \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + az & bx + 2y - z & x + cy + 2z \end{vmatrix} = 0$$

$$\therefore \quad \hat{\imath}\left[\frac{\partial}{\partial y}(x + cy + 2z) - \frac{\partial}{\partial z}(bx + 2y - z)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(x + cy + 2z) - \frac{\partial}{\partial z}(x + y + az)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(bx + 2y - z) - \frac{\partial}{\partial y}(x + y + az)\right] = 0$$

$$\therefore \quad [c - (-1)]\hat{\imath} - [1 - a]\hat{\jmath} + [b - 1]\hat{k} = 0 = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

$$\therefore \quad c + 1 = 0, \quad a - 1 = 0 \quad and \quad b - 1 = 0$$

$$\therefore \quad a = 1, \quad b = 1 \quad and \quad c = -1.$$

HOME WORK:

1. Show that the vector $(-x^2 + yz)\,\hat{\imath} + (4y - z^2x)\,\hat{\jmath} + (2xz - 4z)\,\hat{k}$ is irrotational.
2. Show that $\vec{F} = (y + z)i + (z + x)j + (x + y)k$ is irrotational. Also find a scalar function $\emptyset$ such that $\vec{F} = \nabla\emptyset$.
3. Show that $\vec{F} = (2xy^2 + yz)\hat{\imath} + (2x^2y + xz + 2yz^2)\hat{\jmath} + (2y^2z + xy)\hat{k}$ is irrotational. Also find a scalar function $\emptyset$ such that $\vec{F} = \nabla\emptyset$.
4. Find the values of the a and b such that

   $\vec{F} = (axy + z^3)\,\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3bxz^2 - y)\,\hat{k}$ is irrotational.
5. If $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ and $r = |\vec{r}|$ then prove that $\nabla.(r^n\vec{r}) = (n + 3)r^n$. Hence

   Show that $\frac{\vec{r}}{r^3}$ is solenoidal.

VECTOR INTEGRATION:

If two vectors $\vec{F}(t)$ and $\vec{G}(t)$ be such that $\frac{d}{dt}\left[\vec{G}(t)\right] = \vec{F}(t)$, then $\vec{G}(t)$ is called an integral of $\vec{F}(t)$ w. r. t. scalar variable t and we write $\int \vec{F}(t)dt = \vec{G}(t)$.

If $\vec{C}$ is an arbitrary constant and $\frac{d}{dt}\left[\vec{G}(t) + \vec{C}\right] = \vec{F}(t)$, then $\int \vec{F}(t)dt = \vec{G}(t) + \vec{C}$.

This integral is called indefinite integral of $\vec{F}(t)$ and its definite integral is

$\int_a^b \vec{F}(t)dt = \left[\vec{G}(t) + \vec{C}\right]_a^b = \vec{G}(b) - \vec{G}(a)$.

Line Integral:

Definition:

An integral which is evaluated along curve is called a line integral.

If $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ is a continuous vector point function defined at each point P of a curve C and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is the position vector of the point P on the curve C, then

$\int_C \vec{F}.d\vec{r} = \int_C F_1dx + F_2dy + F_3dz$ is called line integral of $\vec{F}$ along the curve C.

If C is a closed curve then, integral sign $\int_C$ is replaced by $\oint_C$.

If $\vec{F}$ represents the force acting along the curve C, then the total work done by $\vec{F}$ is given

by $\int_C \vec{F}.d\vec{r}$.

Problems:

1. If $\vec{F} = 3xy\hat{\imath} - y^2\hat{\jmath}$, then evaluate $\int_c \vec{F}.d\vec{r}$, where C is the curve in the xy - plane given by $y = 2x^2$ from (0, 0) to (1, 2).

Solution:

Since the curve C is in the xy – plane, we have z = 0. ∴ We take the position vector of the point P(x, y) as $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}$ ∴ $d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath}$.

∴ $\int_c \vec{F}.d\vec{r} = \int_c (3xy\hat{\imath} - y^2\hat{\jmath}).(dx\,\hat{\imath} + dy\,\hat{\jmath}) = \int_c (3xydx - y^2dy)$………(1)

Given C is the parabola $y = 2x^2$ from (0, 0) to (1, 2).

$\therefore$ $dy = 4xdx$ and x varies from 0 to 1. Substituting in equation (1) we get,

$\int_c \vec{F}.d\vec{r} = \int_c (3x.2x^2dx - 4x^4.4xdx) = \int_0^1(6x^3dx - 16x^5dx)$

$$= \left[6\frac{x^4}{4} - 16\frac{x^6}{6}\right]_0^1 = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}.$$

2. If $\vec{A} = (3x^2 + 6y)\,\hat{\imath} - 14yz\,\hat{\jmath} + 20xz^2\,\hat{k}$, evaluate $\int_c \vec{A}.d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the path $x = t$, $y = t^2$, $z = t^3$.

Solution:

Given $\vec{A} = (3x^2 + 6y)\hat{\imath} - 14yz\hat{\jmath} + 20xz^2\hat{k}$, $\therefore$ Take $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

$\therefore$ $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$ $\therefore$ $\vec{A}.d\vec{r} = (3x^2 + 6y)dx - 14yz\,dy + 20xz^2dz$….(1)

Given curve C is $x = t$, $y = t^2$, $z = t^3$ from (0, 0, 0) to (1, 1, 1).

$\therefore$ $dx = dt$, $dy = 2tdt$, $dz = 3t^2dt$. Substituting in equation (1) we get,

$\vec{A}.d\vec{r} = (3t^2 + 6t^2)\,dt - 14t^2.t^3.2tdt + 20t.t^6.3t^2dt = (9t^2 - 28t^6 + 60t^9)dt$

When $x = y = z = 0$, $t = 0$ and when $x = y = z = 1$, $t = 1$.

$\therefore$ $\int_c \vec{A}.d\vec{r} = \int_0^1(9t^2 - 28t^6 + 60t^9)dt = \left[9\frac{t^3}{3} - 28\frac{t^7}{7} + 60\frac{t^{10}}{10}\right]_0^1 = 3 - 4 + 6 = 5.$

3. Find the total work done by the force $\vec{F} = 3xy\hat{\imath} - y\hat{\jmath} + 2zx\hat{k}$ in moving a particle around the circle $x^2 + y^2 = 4$.

Solution:

Total work done by the force $\vec{F}$ is given by $W = \int_c \vec{F}.d\vec{r}$.

Given $\vec{F} = 3xy\,\hat{\imath} - y\hat{\jmath} + 2zx\,\hat{k}$. $\therefore$ Take $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\hat{k}$

$\therefore$ $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$ $\therefore$ $\vec{F}.\,d\vec{r} = 3xydx - y\,dy + 2zx\,dz$……(1)

The parametric equation of the circle $x^2 + y^2 = 4$ is given by

$x = 2\cos\theta$, $y = 2\sin\theta$ and $z = 0$, where $0 \le \theta \le 2\pi$.

$\therefore$ $dx = -2\sin\theta d\theta$, $dy = 2\cos\theta d\theta$, $dz = 0$. Substituting in equation (1) we get,

$\therefore$ $\vec{F}.\,d\vec{r}$ = 3(2cosθ)(2sinθ)($-$ 2 $sin\theta d\theta$) $-$ $(2sin\theta)(2cos\theta d\theta) + 0$

$\therefore$ $\vec{F}.\,d\vec{r}$ = (- 24$sin^2\theta cos\theta$ - 4 $sin\theta cos\theta$)$d\theta$

$\therefore$ W = $\int_c \vec{F}.d\vec{r}$ = $\int_0^{2\pi}(-24sin^2\theta cos\theta - 4\,sin\theta cos\theta)d\theta$

$\therefore$ W = $\left[-24\,\frac{sin^3\theta}{3} - 4\frac{sin^2\theta}{2}\right]_0^{2\pi}$ = 0. Using $\int[f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}$.

HOME WORK:

1. If $\vec{F} = (5xy - 6x^2)\hat{\imath} + (2y - 4x)\hat{\jmath}$, then evaluate $\int_c \vec{F}.\vec{dr}$, where C is the curve in the xy-plane given by y = x^3 from (1, 1) to (2, 8).

2. A vector field is given by $\vec{F} = \;siny\,\hat{\imath} - x(1 + cosy)\hat{\jmath}$. Evaluate the line integral over a circular path given by $x^2 + y^2 = a^2$, z = 0.

3. Find the work done in moving a particle in the force field. $\vec{F} = 3x^2\,\hat{\imath} + (2xz - y)\hat{\jmath} + z\,\hat{k}$, along (i) the straight line from (0,0,0) to (2,1,3). (ii) the curve defined by x = 4y, $3x^3$ = 8z from x = 0 to x = 2.

Surface Integral:

The surface integral of a vector function $\vec{F}$ over a surface S is defined as the integral of the normal component of $\vec{F}$ taken over the surface S.

If $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ is a vector point function over a surface S and $\hat{n}$ is the outward unit normal to the surface S at a point P, then the surface integral of $\vec{F}$ over S is denoted by $\int_S \vec{F}.\vec{ds} = \int_S \vec{F}.\hat{n}\,ds$. i.e., $\iint_S \vec{F}.\vec{ds} = \iint_S \vec{F}.\hat{n}\,ds$. Where $ds = dx.dy$.

Green's Theorem in the plane:

Statement:

If M$(x,y)$and N(x,y) be two continuous functions of x and y having continuous partial derivatives $\frac{\partial N}{\partial x}$ *and* $\frac{\partial M}{\partial y}$ in a region R of xy-plane bounded by a closed curve C then,

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

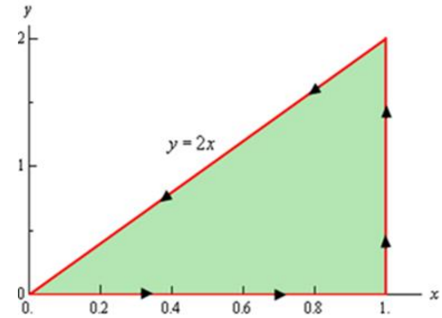
Problems on Greens Theorem:

1. Use Green's Theorem to evaluate $\int_c xydx + x^2y^3dy$, where C is the triangle with vertices (0,0), (1,0), (1,2) with positive orientation.

Solution:

Here M = xy, N = x^2y^3 $\therefore$ $\frac{\partial M}{\partial y} = x$ and $\frac{\partial N}{\partial x} = 2xy^3$.

By Green's Theorem, we have,



$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$$

$$\therefore \int_c xydx + x^2y^3dy = \iint_R (2xy^3 - x)dx\, dy$$

Equation of the line joining (0,0) and (1,2) is y = 2x. Using

$$(y - y_1) = \frac{(y_2-y_1)}{(x_2-x_1)}(x - x_1).$$

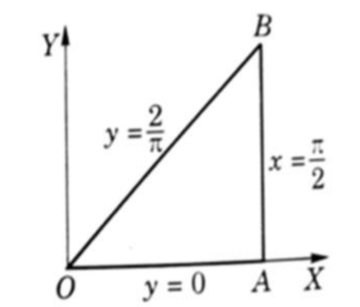
$\therefore$ **x varies from x = 0 to x = 1 and y varies from y = 0 to y = 2x.**

$$\therefore \int_c xydx + x^2y^3dy = \int_0^1 \int_0^{2x} (2xy^3 - x)dydx = \int_0^1 \left[\frac{1}{2}xy^4 - xy\right]_0^{2x} dx$$

$$\therefore \int_c xydx + x^2y^3dy = \int_0^1 (8x^5 - 2x^2)\, dx = \left[\frac{4}{3}x^6 - \frac{2}{3}x^3\right]_0^1 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

2. Using Green's theorem to evaluate $\int_c [(y - sinx)dx + cosx\, dy]$ where C is the plane triangle enclosed by the lines y = 0, x = $\pi/2$ and y = $2x/\pi$.

Solution:



Here M = $y - sinx$, N = $cosx$ $\therefore$ $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = -sinx$

By Green's Theorem, we have,

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$$

$$\therefore \int_c [(y - sinx)dx + cosx\, dy] = \iint_R (-sinx - 1)dy\, dx$$

Here x varies from x = 0 to x = $\pi/2$ and y varies from y = 0 to y = $2x/\pi$.

$\therefore \int_c [(y - sinx)dx + cosx\, dy] = \int_0^{\pi/2}\int_0^{2x/\pi}(-sinx - 1)dydx$

$\therefore \int_c [(y - sinx)dx + cosx\, dy] = -\int_0^{\pi/2}[sinx\,(y) + y]_0^{2x/\pi}\, dx$

$= -\int_0^{\pi/2}[(sinx + 1)y]_0^{2x/\pi}\, dx = -\int_0^{\pi/2}[sinx + 1][2x/\pi]\, dx$

$= -\frac{2}{\pi}\int_0^{\pi/2} x[sinx + 1]\, dx = -\frac{2}{\pi}[x\,(-cosx + x) - 1(-sinx + x^2/2)]_0^{\pi/2}$

$= -\frac{2}{\pi}\left\{\frac{\pi}{2}\left(0 + \frac{\pi}{2}\right) - \left(-1 + \frac{\pi^2}{8}\right)\right\} = -\frac{2}{\pi}\left\{\frac{\pi^2}{4} + 1 - \frac{\pi^2}{8}\right\} = -\frac{2}{\pi}\left(1 + \frac{\pi^2}{8}\right) = -\left(\frac{2}{\pi} + \frac{\pi}{4}\right)$

3. Apply Green's theorem to evaluate $\int_c [(2x^2 - y^2)dx + (x^2 + y^2)\, dy]$ where C is the boundary of the area enclosed by the x-axis and the upper half of the circle $x^2 + y^2 = a^2$.

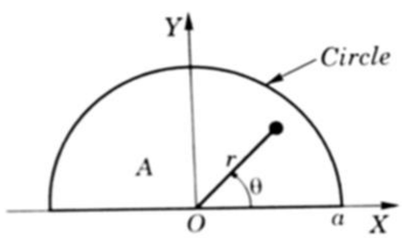
Solution:

Here M = $2x^2 - y^2$, N = $x^2 + y^2$ $\therefore$ $\frac{\partial M}{\partial y}$ = - $2y$ and $\frac{\partial N}{\partial x}$ = $2x$.

By Green's Theorem, we have, $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$

$\therefore \int_c [(2x^2 - y^2)dx + (x^2 + y^2)\, dy] = \iint_R (2x + 2y)dx\, dy = 2\iint_R (x + y)dx\, dy$

Here region R is the upper half of the circle $x^2 + y^2 = a^2$.



Changing into polar coordinates, we take,

$x = r\,cos\theta,\ y = r\,sin\theta.$ $\therefore$ $dx\,dy = rdrd\theta,$ r varies from $r = 0$ to $r = a$ and θ varies from $\theta = 0$ to $\theta = \pi$.

$\therefore \int_c [(2x^2 - y^2)dx + (x^2 + y^2)\, dy] = 2\int_0^{\pi}\int_0^{a}(r\,cos\theta + r\,sin\theta)\, r\, dr\, d\theta$

$= 2\int_0^{\pi}\int_0^{a}(cos\theta + sin\theta)r^2\, dr\, d\theta = 2\left[\int_0^{\pi}(cos\theta + sin\theta)\left[\frac{r^3}{3}\right]_0^a d\theta\right]$

$= 2\left[\int_0^{\pi}(cos\theta + sin\theta)\left(\frac{a^3}{3}\right) d\theta\right] = 2\left(\frac{a^3}{3}\right)[sin\,\theta - cos\theta]_0^{\pi}$

$= \frac{2a^3}{3}[(0 + 0) - (-1 - 1)] = \frac{4a^3}{3}.$

HOME WORK:

1. Use Green's Theorem to evaluate $\int_c [x^2 y dx + x^2 dy]$ where C is the boundary described

   counter clockwise of triangle with vertices (0,0) , (1,0) , (1,1).

2. Evaluate $\int_c [(xy + y^2)dx + x^2 dy]$, where C is bounded by y = x and y = x^2, using

   Green's Theorem.

3. If C is a simple closed curve in the xy-plane not enclosing the origin, show that

   $\int_c \vec{F}.\overrightarrow{dr} = 0$, where $\vec{F} = \frac{y\,\hat{\imath} - x\hat{\jmath}}{x^2+y^2}$, using Green's Theorem.

4. Using Green's theorem evaluate $\int_c [(3x - 8y^2)dx + (4y - 6xy)\, dy]$, where C is the

   boundary of the region bounded by x = 0, y = 0 and x + y = 1.

5. Using Green's theorem evaluate $\int_c [(3x - 8y^2)dx + (4y - 6xy)\, dy]$, where C is the

   boundary of the region bounded by x = 0, y = 0 and x + y = 1.

Stoke's Theorem:

Statement:

If $\vec{F} = f_1\,\hat{\imath} + f_2\,\hat{\jmath} + f_3\,\hat{k}$ is a continuous differential vector point function in the surface S bounded by a simple closed curve C, then $\int_C \vec{F}.d\vec{r} = \iint_S curl\vec{F}.\hat{n}ds$ or $\int_C \vec{F}.d\vec{r} = \iint_S curl\vec{F}.\hat{n}dxdy$. Where $\hat{n}$ is a unit external normal at any point on S.

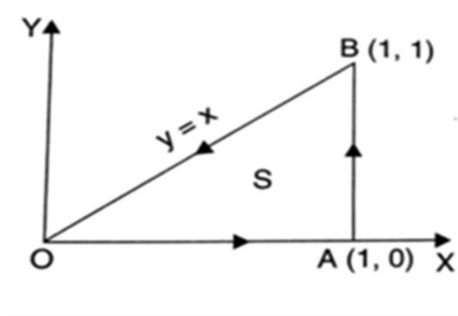
Problems on Stoke's Theorem:

1. Use Stoke's theorem to evaluate $\int_c \vec{F}.d\vec{r}$ where $\vec{F} = y^2\hat{\imath} + x^2\hat{\jmath} - (x+z)\hat{k}$ and

   C is the boundary of the triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0).

Solution:

Given $\vec{F} = y^2\hat{\imath} + x^2\hat{\jmath} - (x+z)\hat{k}$. i.e., $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$



Where $F_1 = y^2$, $F_2 = x^2$, $F_3 = -(x+z)$.

$$\therefore \quad \text{curl}\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$\therefore$ curl $\vec{F} = \hat{\imath}(0-0) - \hat{\jmath}(-1-0) + \hat{k}(2x - 2y)\,\hat{k} = \hat{\jmath} + 2(x-y)\,\hat{k}$

Given C is the boundary of the triangle with vertices at (0, 0, 0), (1,0, 0) and (1, 1, 0).

Here, z-coordinate of each vertex of the triangle is zero.

$\therefore$ *T*he triangle lies in the xy- plane and the unit normal vector to the plane is $\hat{n} = \hat{k}$.

$\therefore$ $curl\vec{F}\,.\,\hat{n} = [\hat{\jmath} + 2(x-y)\,\hat{k}]\,.\,\hat{k} = 2(x-y)$

By Stoke's theorem, we have, $\int_C \vec{F}\,.\,d\vec{r} = \iint_S curl\vec{F}\,.\,\hat{n}dS$.

$\therefore \int_c \vec{F}.\,d\vec{r} = \iint_S 2(x-y)\,dydx.$

The equation of the line joining O(0, 0) and B(1, 1) is y = x.

Using $(y - y_1) = \frac{(y_2-y_1)}{(x_2-x_1)}(x - x_1)$.

$\therefore$ x varies from $x = 0$ to $x = 1$ and y varies from y = 0 to y = x.

$$\therefore \int_c \vec{F}\,.\,d\vec{r} = \int_0^1 \int_0^x 2(x-y)\,dydx = \int_0^1 2\left[xy - \frac{y^2}{2}\right]_0^x dx$$

$$= \int_0^1 2\left(x^2 - \frac{x^2}{2}\right)dx = \int_0^1 x^2dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

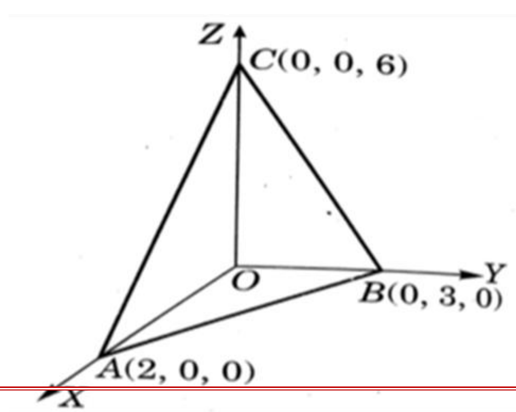
2. Use Stoke's theorem to evaluate $\int_c [(x+y)dx + (2x-z)dy + (y+z)dz]$

where C is the boundary of the triangle with vertices at (2, 0, 0), (0, 3, 0) and (0, 0, 6).

Solution:

Given $\vec{F} = (x+y)\hat{\imath} + (2x-z)\hat{\jmath} + (y+z)\hat{k}$.

i.e., $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$



Where $F_1 = x + y, \quad F_2 = 2x - z, \quad F_3 = y + z.$

$$\therefore \quad \text{Curl } \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$\therefore$ **Curl** $\vec{F} = \hat{\imath}(1+1) - \hat{\jmath}(0-0) + \hat{k}(2-1) = 2\hat{\imath} + \hat{k}.$

Given C is the boundary of the triangle with vertices at

A(2, 0, 0), B(0, 3, 0) and C(0, 0, 6).

$\therefore$ **The equation of the plane passing through A(2, 0, 0), B(0, 3, 0) and C(0, 0, 6) is**

$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1. \quad i.e., 3x + 2y + z = 6$

Consider $\emptyset = 3x + 2y + z - 6 \quad \therefore \quad \nabla\emptyset = \frac{\partial\emptyset}{\partial x}\hat{\imath} + \frac{\partial\emptyset}{\partial y}\hat{\jmath} + \frac{\partial\emptyset}{\partial z}\hat{k} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}.$

$\therefore \quad |\nabla\emptyset| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$

$\therefore$ **the unit normal vector to the plane is** $\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{(3\hat{\imath} + 2\hat{\jmath} + \hat{k})}{\sqrt{14}}$

$\therefore \quad curl\vec{F}.\hat{n} = (2\hat{\imath} + \hat{k}) \, . \, \frac{(3\hat{\imath} + 2\hat{\jmath} + \hat{k})}{\sqrt{14}} = \frac{6+0+1}{\sqrt{14}} = \frac{7}{\sqrt{14}}$

By Stoke's theorem, we have, $\int_C \vec{F} . d\vec{r} = \iint_S curl\vec{F} . \hat{n} \, dS.$

$\therefore \int_c \vec{F}. d\vec{r} = \iint_S \frac{7}{\sqrt{14}} dS = \frac{7}{\sqrt{14}} \iint_S dS = \frac{7}{\sqrt{14}} \iint_S dx\, dy = \frac{7}{\sqrt{14}} (Area\ of\ the\ \Delta\ ABC) \ldots (1)$

Now area of the $\Delta\, ABC = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$

Since the vertices of the triangle are A(2, 0, 0), B(0, 3, 0) and C(0, 0, 6), we get,

$\overrightarrow{OA} = 2\hat{\imath} + 0\hat{\jmath} + 0\hat{k}, \quad \overrightarrow{OB} = 0\hat{\imath} + 3\hat{\jmath} + 0\hat{k}, \quad \overrightarrow{OC} = 0\hat{\imath} - 0\hat{\jmath} + 6\hat{k}.$

$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\hat{\imath} + 3\hat{\jmath} + 0\hat{k}$ **and** $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -2\hat{\imath} + 0\hat{\jmath} + 6\hat{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 3 & 0 \\ -2 & 0 & 6 \end{vmatrix} = \hat{\imath}(18-0) - \hat{\jmath}(-12-0) + \hat{k}(0-6) = 18\,\hat{\imath} + 12\hat{\jmath} - 6\hat{k}$$

$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(18)^2 + (12)^2 + (-6)^2} = \sqrt{504} = \sqrt{36X14} = 6\sqrt{14}$

$\therefore$ **Area** $\Delta\, ABC = \frac{1}{2}\,|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}(6\sqrt{14}) = 3\sqrt{14}$

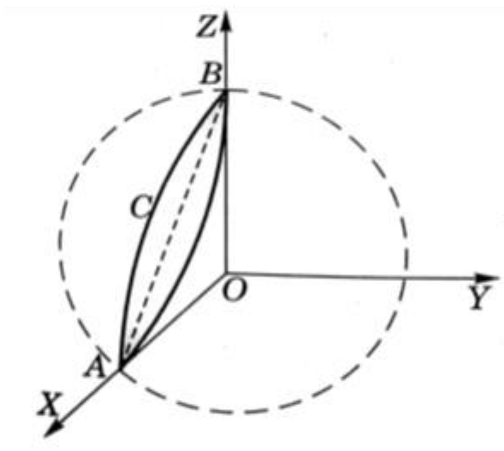
Equation (1) becomes $\int_C \vec{F}\,.\,d\vec{r} = \frac{7}{\sqrt{14}}\left(3\sqrt{14}\right) = 21.$

3. Apply Stoke's theorem to evaluate $\int_c \,[y\,dx + z dy + x\,dz]$ **where C is the curve of intersection of** $x^2 + y^2 + z^2 = a^2$ **and** $x + z = a$.

Solution:

Given $\vec{F} = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$. **i.e.,** $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$

Where $F_1 = y,\quad F_2 = z,\quad F_3 = x.$



$\therefore$ **Curl** $\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$

$= \hat{\imath}(0-1) - \hat{\jmath}(1-0) + \hat{k}(0-1).$

$\therefore$ **Curl** $\vec{F} = -\hat{\imath} - \hat{\jmath} - \hat{k}.$

Given C is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ **and** $x + z = a$. **Therefore the curve C is the circle lies on the plane** $x + z = a$. **i.e.,** $x + z - a = 0$.

Consider $\emptyset = x + z - a$.

$We\ have,\ \nabla\emptyset = \frac{\partial\emptyset}{\partial x}\hat{\imath} + \frac{\partial\emptyset}{\partial y}\hat{\jmath} + \frac{\partial\emptyset}{\partial z}\hat{k}$

$\therefore \nabla\emptyset = \hat{\imath} + 0\hat{\jmath} + \hat{k}.\quad \therefore\ \nabla\emptyset = \hat{\imath} + \hat{k} \quad \therefore\ |\nabla\emptyset| = \sqrt{1+1} = \sqrt{2}$

$\therefore$ **the unit normal vector to the plane is** $\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{\hat{\imath} + \hat{k}}{\sqrt{2}}$

$\therefore\ curl\vec{F}\,.\,\hat{n} = (-\hat{\imath} - \hat{\jmath} - \hat{k})\,.\,\frac{(\hat{\imath}+\hat{k})}{\sqrt{2}} = \frac{(-1+0-1)}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$

By Stoke's theorem, we have, $\int_C \vec{F}\,.\,d\vec{r} = \iint_S curl\vec{F}\,.\,\hat{n}\,dS.$

$\therefore\ \int_C \vec{F}\,.\,d\vec{r} = \iint_S (-\sqrt{2})\,dS = -\sqrt{2}\iint_S dS = -\sqrt{2}\,(Area\ of\ the\ circle)$**……(1)**

Now the equation of the plane $x + z = a$ can be written as $\frac{x}{a} + \frac{z}{a} = 1$.

$\therefore$ The points of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$ are A(a, 0, 0) and B(0, 0, a).

$\therefore$ Diameter AB of the circle = $\sqrt{(a-0)^2 + (0-0)^2 + (0-a)^2} = \sqrt{a^2 + a^2}$

$= \sqrt{2a^2} = \sqrt{2}a.$

$\therefore$ Radius of the circle = r = $\frac{Diameter}{2} = \frac{\sqrt{2}\,a}{2} = \frac{a}{\sqrt{2}}$

$\therefore$ Area of the circle = $\pi r^2 = \pi\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{\pi a^2}{2}$

Equation (1) becomes $\int_C \vec{F}.d\vec{r} = -\sqrt{2}\left(\frac{\pi a^2}{2}\right) = \frac{-\pi a^2}{\sqrt{2}}$.

HOME WORK:

1. Use Stoke's theorem to evaluate $\int_c \vec{F}.\overrightarrow{dr}$ where $\vec{F} = (x^2 + y^2)\hat{\imath} - 2xy\hat{\jmath}$ and C is the Rectangle bounded by the lines x = $\pm$ a, y = 0, y = b.
2. Use Stoke's theorem to evaluate $\int_c \vec{F}.\overrightarrow{dr}$ where $\vec{F} = (2x - y)\hat{\imath} - yz^2\hat{\jmath} - y^2 z\,\hat{k}$ and C is upper half of the surface of $x^2 + y^2 + z^2 = 1$, bounded by its projection on the xy-plane.
3. If $\vec{F} = 3y\,\hat{\imath} - xz\hat{\jmath} + yz^2\,\hat{k}$ and S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by z = 2, evaluate $\iint_S (\nabla \times \vec{F}).\overrightarrow{ds}$ using Stoke's theorem.

Glance:

1. Gradient of $\emptyset(x, y, z) = c$
   $\text{grad}\emptyset = \nabla\emptyset = \frac{\partial\emptyset}{\partial x}\hat{\imath} + \frac{\partial\emptyset}{\partial y}\hat{\jmath} + \frac{\partial\emptyset}{\partial z}\hat{k}$
2. Unit normal vector of $\emptyset(x, y, z) = c$
   $\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|}$
3. Directional derivative of $\emptyset$ along $\vec{a} = \nabla\emptyset\,.\,\hat{a}$

4. **Maximum directional derivative** $= |\nabla \emptyset|$
5. **Angle between the surfaces** $\cos\theta = \frac{\nabla\emptyset_1 . \nabla\emptyset_2}{|\nabla\emptyset_1||\nabla\emptyset_2|}$
6. **Divergence of a vector** $\vec{F} = F_1\,\hat{\imath} + F_2\,\hat{\jmath} + F_3\,\hat{k}$

   $$div\,\vec{F} = \nabla.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$
7. **Solenoidal :** $div\,\vec{F} = \nabla.\vec{F} = 0$
8. **Curl of vector** $\vec{F} = F_1\,\hat{\imath} + F_2\,\hat{\jmath} + F_3\,\hat{k}$

   $$\text{Curl}\,\vec{F} = \nabla\,X\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$
9. **Irrotational: Curl** $\vec{F} = \nabla\,X\,\vec{F} = 0$
10. **Line integral** $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$

    $$\int_C \vec{F}.d\vec{r} = \int_C F_1dx + F_2dy + F_3dz$$
11. **By Green's Theorem,**

    $$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$
12. **By Stoke's theorem,**

    $$\int_C \vec{F}\,.d\vec{r} = \iint_S curl\vec{F}\,.\hat{n}dS. = = \iint_S curl\vec{F}\,.\hat{n}dxdy$$